Some basic cubic splines

Alternate formulation

Consider these piecewise cubic functions:

$$f(x) = \begin{cases} \text{undefined} & \text{if } \quad x \le 0 \\ x^3 - 6x + 6 & \text{if } 0 \le x \le 1 \\ (2-x)^3 \;=\; -x^3 + 6x^2 - 12x + 8 & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 2 \le x \end{cases} \tag{1}$$

$$g(x) = \begin{cases} \text{undefined} & \text{if } \quad x \le 0 \\ 6x - 2x^3 & \text{if } 0 \le x \le 1 \\ 3x^3 - 15x^2 + 21x - 5 & \text{if } 1 \le x \le 2 \\ (3-x)^3 \;=\; -x^3 + 9x^2 - 27x + 27 & \text{if } 2 \le x \le 3 \\ 0 & \text{if } 3 \le x \end{cases} \tag{3}$$

$$h(x) = \begin{cases} 0 & \text{if } \quad x \le 0 \\ x^3 & \text{if } 0 \le x \le 1 \\ -3x^3 + 12x^2 - 12x + 4 & \text{if } 1 \le x \le 2 \\ 3x^3 - 24x^2 + 60x - 44 & \text{if } 2 \le x \le 3 \\ (4-x)^3 \;=\; -x^3 + 12x^2 - 48x + 64 & \text{if } 3 \le x \le 4 \\ 0 & \text{if } 4 \le x \end{cases} \tag{6}$$

The reader is invited to check that each of f, g and h is a cubic spline where it is defined. (That is, h is defined on the whole real line; f and g are defined for the non-negative real axis $x \ge 0$.) To focus on h as an example, it is necessary to prove the following: at each point of overlap (namely 0, 1, 2, 3 and 4), the two cubic polynomials that meet at that overlap point should agree, both in their function values and the values of their first and second derivatives. It may be tedious to verify this fact, but the job needs to be done only once. Once this fact is verified, these can be kept in a library for use over and over. (Translation to computerese: make three subroutines: one to calculate f, one for g and one for h.)

The next thing to notice is that one can shift these around. Let us for definiteness consider the interval $[0,6]$. Then we can define seven different spline curves:

$$\begin{aligned}
f_0(x) &= f(x) && (7)\\
f_1(x) &= g(x) && (8)\\
f_2(x) &= h(x) && (9)\\
f_3(x) &= h(x-1) && (10)\\
f_4(x) &= h(x-2) && (11)\\
f_5(x) &= g(6-x) && (12)\\
f_6(x) &= f(6-x) && (13)
\end{aligned}$$

As one can easily check, the first row of the following matrix show $f_i(0)$ for $i = 0, \ldots\ 6$, the next row shows $f_i(1)$, and so on.

$$\begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 4 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 4 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 4 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 4 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 4 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}$$

Therefore, if we want a spline $p(x)$ that satisfies $p(j) = a_j$, for $j = 0, \ldots\ 6$, all we have to do is solve the equations

$$\begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 4 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 4 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 4 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 4 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 4 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\begin{bmatrix} u_0\\ u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6 \end{bmatrix}
=
\begin{bmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6 \end{bmatrix}, \tag{14}$$

and then define

$$p(x) = \sum_{i=0}^{6} u_i f_i(x) \tag{15}$$

Now suppose that we wish to find the spline that interpolates 1 and -1 alternately, in other words, the spline that has the initial condition

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}. \tag{16}$$

To solve for the unknown coefficients u_i, we first invert the 7×7 matrix B in (14) yielding B^{-1} =

$$\begin{bmatrix} 0.1667 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0447 & 0.2679 & -0.0718 & 0.0192 & -0.0051 & 0.0013 & -0.0002 \\ 0.0120 & -0.0718 & 0.2872 & -0.0769 & 0.0205 & -0.0051 & 0.0009 \\ -0.0032 & 0.0192 & -0.0769 & 0.2885 & -0.0769 & 0.0192 & -0.0032 \\ 0.0009 & -0.0051 & 0.0205 & -0.0769 & 0.2872 & -0.0718 & 0.0120 \\ -0.0002 & 0.0013 & -0.0051 & 0.0192 & -0.0718 & 0.2679 & -0.0447 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1667 \end{bmatrix}$$

Equation (14) is now solved as

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = B^{-1} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1667 \\ -0.4103 \\ 0.4744 \\ -0.4872 \\ 0.4744 \\ -0.4103 \\ 0.1667 \end{bmatrix} \tag{17}$$

Thus the spline that satisfies our initial condition (16) can be described as the following special case of Equation (15):

$$\begin{aligned} p(x) \quad = \quad & 0.1667\, f_0(x) \;-\; 0.4103\, f_1(x) \;+\; 0.4744\, f_2(x) \;-\; 0.4872\, f_3(x) \\ & + 0.4744\, f_4(x) \;-\; 0.4103\, f_5(x) \;+\; 0.1667\, f_6(x) \end{aligned} \tag{18}$$

The way this works, as always, is that we have to calculate this sum for each interval separately. Let's begin with the interval from 0 to 1. On this interval, f_3, f_4, f_5 and f_6 are zero, so the sum reduces to

$$p(x) \quad = \quad 0.1667\, f_0(x) \;-\; 0.4103\, f_1(x) \;+\; 0.4744\, f_2(x) \tag{19}$$

$$\quad = \quad 1.4617\, x^3 \;-\; 3.4617\, x \;+\; 1.0000 \tag{20}$$

(for $0 \leq x \leq 1$). At this point you can readily confirm that $p(0) = 1$ and $p(1) = -1$. Of course $p(2)$ will have to wait.

We leave to the reader the calculation of the coefficients of p for the other subintervals. In practice,[1] these would not be calculated as such. Rather, one has subroutines for calculating $f_0(x)$, ... , $f_6(x)$, and then one further subroutine to effect the sum (18) (general case in (15)).

[1] Well, my practice anyway. I always sacrifice small amounts of calculation time for clarity of presentation in the algorithm. If $p(x)$ were to be used innumerable times with the same coefficients, then it might be helpful to drop the subroutine approach.