# UNIPOTENT HECKE ALGEBRAS: the structure, representation theory, and combinatorics

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# Abstract


**Unipotent Hecke algebras**

The Iwahori-Hecke algebra and Gelfand-Graev Hecke algebras have been critical in developing the representation theory of finite groups and its implications in algebraic combinatorics. This thesis lays the foundations for a theory of unipotent Hecke algebras, a family of Hecke algebras that includes both the classical Gelfand-Graev Hecke algebra and a generalization of the Iwahori-Hecke algebra (the Yokonuma Hecke algebra). In particular, this includes a combinatorial analysis of their structure and representation theory. My main results are (a) a braid-like multiplication algorithm for a natural basis; (b) a canonical commutative subalgebra that may be used for weight space decompositions of irreducible modules. When the underlying group is the general linear group $GL_n(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ with $q$ elements, I also supply (c) an explicit construction of the natural basis; (d) a generalization of a Robinson-Schensted-Knuth correspondence (a bijective proof of representation theoretic identities); and (e) an explicit construction of the irreducible Yokonuma algebra modules using the known irreducible module structure of the Iwahori-Hecke algebra.

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# Contents

# Chapter 1

# Introduction

Combinatorial representation theory takes abstract algebraic structures and attempts to give explicit tools for working with them. While the central questions may take an abstract form, the answers from this perspective use known combinatorial objects in computations and constructive descriptions. This thesis examines a family of Hecke algebras, giving algorithms that elucidate some of the structural and representation theoretic results classically developed from an algebraic and geometric point of view. Thus, a classical description of a natural basis becomes an actual construction of this basis, and an efficient algorithm for computing products in this basis replaces a classical formula.

A large body of literature in combinatorial representation theory is dedicated to the study of the symmetric group $S_n$, giving rise to a wealth of combinatorial ideas. Some of these ideas have been generalized to Weyl groups, a larger class of groups whose structure is essential in the study of Lie groups and Lie algebras. Even more generally, the Iwahori-Hecke algebra gives a “quantum” version of the Weyl group situation.

Iwahori [Iwa64] and Iwahori-Matsumoto [IM65] introduced the Iwahori-Hecke algebra as a first step in classifying the irreducible representations of finite Chevalley groups and reductive $p$-adic Lie groups. Subsequent work (e.g. [CS99] [KL79] [LV83]) has established Hecke algebras as fundamental tools in the representation theory of Lie groups and Lie algebras, and advances on subfactors and quantum groups by Jones [Jon83], Jimbo [Jim86], and Drinfeld [Dri87] gave Hecke algebras a central role in knot theory [Jon87], statistical mechanics [Jon89], mathematical physics, and operator algebras.

Unipotent Hecke algebras are a family of algebras that generalize the Iwahori-Hecke algebra. Another classical example of a unipotent Hecke algebra is the Gelfand-Graev Hecke algebra, which has connections with unipotent orbits [Kaw75], Kloosterman sums [CS99], and in the $p$-adic case, Hecke operators (see [Bum98, 4.6,4.8]). Interpolating between these two classical examples will be fundamental to my analysis.

There are three ingredients in the construction of a Hecke algebra: a group $G$, a subgroup $U$ and a character $\chi : U \to \mathbb{C}$. The Hecke algebra associated with the triple $(G, U, \chi)$ is the centralizer algebra

$$\mathcal{H}(G, U, \chi) = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\chi)).$$

It has a natural basis indexed by a subset $N_\chi$ of $U$-double coset representatives in $G$.

A unipotent Hecke algebra is

$$\mathcal{H}_\mu = \mathcal{H}(G, U, \psi_\mu) = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\psi_\mu)),$$

where $U$ is a maximal unipotent subgroup of a finite Chevalley $G$ (such as $GL_n(\mathbb{F}_q)$), and $\psi_\mu$ is an arbitrary linear character. In this context, the classical algebras are

$$\begin{array}{rcl} \text{the Yokonuma Hecke algebra [Yok69b]} & \longleftrightarrow & \psi_\mu \text{ trivial,} \\ \text{the Gelfand-Graev Hecke algebra [GG62]} & \longleftrightarrow & \psi_\mu \text{ in general position.} \end{array}$$

After reviewing some of the underlying mathematics in Chapter 2, Chapter 3 analyzes unipotent Hecke algebras of general type. The main results are

(a) a generalization to unipotent Hecke algebras of the braid calculus used in Iwahori-Hecke algebra multiplication. I use two approaches: the first gives "local" relations similar to braid relations, and the second gives a more global algorithm for computing products. In particular, this provides another solution to the difficult problem of multiplication in the Gelfand-Graev Hecke algebra (see [Cur88] for a geometric approach to this problem).

(b) the construction of a large commutative subalgebra $\mathcal{L}_\mu$ in $\mathcal{H}_\mu$, which may be used as a Cartan subalgebra to analyze the representations of $\mathcal{H}_\mu$ via weight space decompositions;

Chapters 4, 5 and part of 6 consider the case when $G = GL_n(\mathbb{F}_q)$, the general linear group over a finite field with $q$ elements. These chapters can all be read independently of Chapter 3, and I would recommend that readers unfamiliar with Chevalley groups begin with Chapter 4.

Chapter 4 describes the structure of the unipotent Hecke algebras when $G = GL_n(\mathbb{F}_q)$; in this case, unipotent Hecke algebras are indexed by compositions $\mu$ of $n$. The main results are

(c) an explicit basis. Suppose $\mathcal{H}_\mu$ corresponds to the composition $\mu$, and $N_\mu$ indexes the basis of $\mathcal{H}_\mu$. There is an explicit bijection from $N_\mu$ to the set of polynomial matrices over $\mathbb{F}_q$ with row-degree sums and column-degree sums equal to $\mu$ (see Chapter 4 for details). In particular, this gives an explicit construction of the matrices in $N_\mu$ .

(d) a pictorial approach to multiplying the natural basis elements, inspired by (a) above.

Chapter 5 proceeds to develop the representation theory of unipotent Hecke algebras in the case $G = GL_n(\mathbb{F})$. This chapter relies heavily on a decomposition of $\mathrm{Ind}_U^G(\psi_\mu)$ given by Zelevinsky in [Zel81]. The main results are

(e) a generalization of the RSK correspondence which provides a combinatorial proof of the identity

$$\dim(\mathcal{H}_\mu) = \sum_{\substack{\text{Irreducible} \\ \mathcal{H}_\mu\text{-modules } V}} \dim(V)^2;$$

(f) a computation of the weight space decomposition of an $\mathcal{H}_\mu$-module with respect to a commutative subalgebra $\mathcal{L}_\mu$ (see (b) above).

Chapter 6 concludes by studying the representation theory of Yokonuma Hecke algebras (the special case when $\psi_\mu$ is trivial). The main results are

(g) a reduction in general type from the study of irreducible Yokonuma Hecke algebra modules to the study of irreducible modules for a family of “sub”algebras;

(h) an explicit construction of the irreducible modules of the Yokonuma algebra in the case $G = GL_n(\mathbb{F}_q)$.

# Chapter 2

# Preliminaries

## 2.1 Hecke algebras

This section gives some of the main representation theoretic results used in this thesis, and defines Hecke algebras. Three roughly equivalent structures are commonly used to describe the representation theory of an algebra $A$:

**(1) Modules.** Modules are vector spaces on which $A$ acts by linear transformations.

**(2) Representations.** Every choice of basis in an $n$-dimensional vector space on which $A$ acts gives a representation, or an algebra homomorphism from $A$ to the algebra of $n \times n$ matrices.

**(3) Characters.** The trace of a matrix is the sum of its diagonal entries. A character is the composition of the algebra homomorphism of (2) with the trace map.

The following discussion will focus on modules and characters.

### 2.1.1 Modules and characters

Let $A$ be a finite dimensional algebra over the complex numbers $\mathbb{C}$. An *$A$-module $V$* is a finite dimensional vector space over $\mathbb{C}$ with a map

$$\begin{array}{ccc} A \times V & \longrightarrow & V \\ (a, v) & \mapsto & av \end{array}$$

such that

$$\begin{aligned} (c1)v &= cv, && c \in \mathbb{C}, \\ (ab)v &= a(bv), && a, b \in A, v \in V, \\ (a+b)(c_1v_1 + c_2v_2) &= c_1av_1 + c_2av_2 + c_1bv_1 + c_2bv_2, && c_1, c_2 \in \mathbb{C}. \end{aligned}$$

An $A$-module $V$ is *irreducible* if it contains no nontrivial, proper subspace $V'$ such that $av' \in V'$ for all $a \in A$ and $v' \in V'$.

An *$A$-module homomorphism* $\theta : V \to V'$ is a $\mathbb{C}$-linear transformation such that

$$a\theta(v) = \theta(av), \qquad \text{for } a \in A, v \in V,$$

and an *A-module isomorphism* is a bijective $A$-module homomorphism. Write

$$\begin{aligned}\mathrm{Hom}_A(V,V') &= \{A\text{-module homomorphisms } V \to V'\}\\ \mathrm{End}_A(V) &= \mathrm{Hom}_A(V,V).\end{aligned}$$

Let $\hat{A}$ be an indexing set for the irreducible modules of $A$ (up to isomorphism), and fix a set

$$\{A^\gamma\}_{\gamma\in\hat{A}}$$

of isomorphism class representatives. An algebra $A$ is *semisimple* if every $A$-module $V$ decomposes

$$V \cong \bigoplus_{\gamma\in\hat{A}} m_\gamma(V)A^\gamma, \qquad \text{where} \quad m_\gamma(V) \in \mathbb{Z}_{\geq 0}, \quad m_\gamma(V)A^\gamma = \underbrace{A^\gamma \oplus \cdots \oplus A^\gamma}_{m_\gamma(V)\text{ terms}}.$$

In this thesis, all algebras (though not necessarily all Lie algebras) are semisimple.

**Lemma 2.1 (Schur's Lemma).** *Suppose $\gamma, \mu \in \hat{A}$ with corresponding irreducible modules $A^\gamma$ and $A^\mu$. Then*

$$\dim(\mathrm{Hom}_A(A^\gamma, A^\mu)) = \delta_{\gamma\mu} = \begin{cases} 1, & \text{if } \gamma = \mu,\\ 0, & \text{otherwise.}\end{cases}$$

Suppose $A \subseteq B$ is a subalgebra of $B$ and let $V$ be an $A$-module. Then $B \otimes_A V$ is the vector space over $\mathbb{C}$ presented by generators $\{b \otimes v \mid b \in B, v \in V\}$ with relations

$$\begin{aligned} ba \otimes v &= b \otimes av, & a \in A, b \in B, v \in V,\\ (b_1 + b_2) \otimes (v_1 + v_2) &= b_1 \otimes v_1 + b_1 \otimes v_2 + b_2 \otimes v_1 + b_2 \otimes v_2, & b_1, b_2 \in B, v_1, v_2 \in V.\end{aligned}$$

Note that the map

$$\begin{array}{ccc} B \times (B \otimes_A V) & \longrightarrow & B \otimes_A V\\ (b', b \otimes v) & \mapsto & (b'b) \otimes v\end{array}$$

makes $B \otimes_A V$ a $B$-module. Define *induction* and *restriction* (respectively) as the maps

$$\begin{array}{cccc} \mathrm{Ind}_A^B : & \{A\text{-modules}\} & \longrightarrow & \{B\text{-modules}\}\\ & V & \mapsto & B \otimes_A V\\ \mathrm{Res}_A^B : & \{B\text{-modules}\} & \longrightarrow & \{A\text{-modules}\}\\ & V & \mapsto & V \end{array}.$$

Suppose $V$ is an $A$-module. Every choice of basis $\{v_1, v_2, \ldots, v_n\}$ of $V$ gives rise to an algebra homomorphism

$$\rho : A \to M_n(\mathbb{C}), \qquad \text{where } M_n(\mathbb{C}) = \{n \times n \text{ matrices with entries in } \mathbb{C}\}.$$

The *character* $\chi_V : A \to \mathbb{C}$ associated to $V$ is

$$\chi_V(a) = \mathrm{tr}(\rho(a)).$$

The character $\chi_V$ is independent of the choice of basis, and if $V \cong V'$ are isomorphic $A$-modules, then $\chi_V = \chi_{V'}$. A character $\chi_V$ is *irreducible* if $V$ is irreducible. The *degree* of a character $\chi_V$ is $\chi_V(1) = \dim(V)$, and if $\chi(1) = 1$, then a character is *linear*; note that linear characters are both irreducible characters and algebra homomorphisms.

Define

$$R[A] = \mathbb{C}\text{-span}\{\chi^\gamma \mid \gamma \in \hat{A}\}, \qquad \text{where } \chi^\gamma = \chi_{A^\gamma},$$

with a $\mathbb{C}$-bilinear inner product given by

$$\langle \chi^\gamma, \chi^\mu \rangle = \delta_{\gamma\mu}. \tag{2.1}$$

Note that the semisimplicity of $A$ implies that any character is contained in the subspace

$$\mathbb{Z}_{\geq 0}\text{-span}\{\chi^\gamma \mid \gamma \in \hat{A}\}.$$

If $\chi_V$ is a character of a subalgebra $A$ of $B$, then let

$$\mathrm{Ind}_A^B(\chi_V) = \chi_{B \otimes_A V}$$

and if $\chi_{V'}$ is a character of $B$, then $\mathrm{Res}_A^B(\chi_{V'})$ is the restriction of the map $\chi_{V'}$ to $A$.

**Proposition 2.2 (Frobenius Reciprocity).** *Let $A$ be a subalgebra of $B$. Suppose $\chi$ is a character of $A$ and $\chi'$ is a character of $B$. Then*

$$\langle \mathrm{Ind}_A^B(\chi), \chi' \rangle = \langle \chi, \mathrm{Res}_A^B(\chi') \rangle.$$

### 2.1.2 Weight space decompositions

Suppose $A$ is a commutative algebra. Then

$$\dim(A^\gamma) = 1, \qquad \text{for all } \gamma \in \hat{A}.$$

The corresponding character $\chi^\gamma : A \to \mathbb{C}$ is an algebra homomorphism, and we may identify the label $\gamma$ with the character $\chi^\gamma$, so that

$$av = \chi^\gamma(a)v = \gamma(a)v, \qquad \text{for } a \in A,\ v \in A^\gamma.$$

If $A$ is a commutative subalgebra of $B$ and $V$ is an $B$-module, then as a $A$-module

$$V = \bigoplus_{\gamma \in \hat{A}} V_\gamma, \qquad \text{where } V_\gamma = \{v \in V \mid av = \gamma(a)v,\ a \in A\}.$$

The subspace $V_\gamma$ is the $\gamma$-*weight space* of $V$, and if $V_\gamma \neq 0$ then $V$ *has a weight* $\gamma$.

For large subalgebras $A$, such a decomposition can help construct the module $V$. For example,

1. If $0 \neq v_\gamma \in V_\gamma$, then $\{v_\gamma\}_{V_\gamma \neq 0}$ is a linearly independent set of vectors. In particular, if $\dim(V_\gamma) = 1$ for all $V_\gamma \neq 0$, then $\{v_\gamma\}$ is a basis for $V$.

2. If $V$ is irreducible, then $BV_\gamma = V$. Chapter 6 uses this idea to construct irreducible modules of the Yokonuma algebra.

### 2.1.3 Characters in a group algebra

Let $G$ be a finite group. The *group algebra* $\mathbb{C}G$ is the algebra

$$\mathbb{C}G = \mathbb{C}\text{-span}\{g \in G\}$$

with basis $G$ and multiplication determined by the group multiplication in $G$. A *$G$-module* $V$ is a vector space over $\mathbb{C}$ with a map

$$\begin{array}{ccc} G \times V & \longrightarrow & V \\ (g, v) & \mapsto & gv \end{array}$$

such that

$$(gg')v = g(g'v), \quad g(cv + c'v') = cgv + c'gv', \quad \text{for } c, c' \in \mathbb{C}, g, g' \in G, v, v' \in V.$$

Note that there is a natural bijection

$$\begin{array}{ccc} \{\mathbb{C}G\text{-modules}\} & \longleftrightarrow & \{G\text{-modules}\} \\ V & \mapsto & V \end{array},$$

so I will use $G$-modules and $\mathbb{C}G$-modules interchangeably. Let $\hat{G}$ index the irreducible $\mathbb{C}G$-modules.

The inner product (2.1) on $R[G]$ has an explicit expression

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & R[G] \times R[G] & \longrightarrow & \mathbb{C} \\ & (\chi, \chi') & \mapsto & \displaystyle\frac{1}{|G|} \sum_{g \in G} \chi(g)\chi'(g^{-1}), \end{array}$$

and if $U$ is a subgroup of $G$, then induction is given by

$$\begin{array}{rccc} \mathrm{Ind}_U^G(\chi) : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \displaystyle\frac{1}{|U|} \sum_{\substack{x \in G \\ xgx^{-1} \in U}} \chi(xgx^{-1}). \end{array}$$

### 2.1.4 Hecke algebras

An *idempotent* $e \in \mathbb{C}G$ is a nonzero element that satisfies $e^2 = e$. For $\gamma \in \hat{G}$, let

$$e_\gamma = \frac{\chi^\gamma(1)}{|G|} \sum_{g \in G} \chi^\gamma(g^{-1}) g \quad \in \mathbb{C}G.$$

These are the *minimal central idempotents* of $\mathbb{C}G$, and they satisfy

$$1 = \sum_{\gamma \in \hat{G}} e_\gamma, \qquad e_\gamma^2 = e_\gamma, \qquad e_\gamma e_\mu = \delta_{\gamma\mu} e_\gamma, \qquad \mathbb{C}G e_\gamma \cong \dim(G^\gamma) G^\gamma.$$

In particular, as a $G$-module,

$$\mathbb{C}G \cong \bigoplus_{\gamma \in \hat{G}} \mathbb{C}G e_\gamma.$$

Let $U$ be a subgroup of $G$ with an irreducible $U$-module $M$. The *Hecke algebra* $\mathcal{H}(G, U, M)$ is the centralizer algebra

$$\mathcal{H}(G, U, M) = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(M)).$$

If $e_M \in \mathbb{C}U$ is an idempotent such that $M \cong \mathbb{C}U e_M$, then

$$\mathrm{Ind}_U^G(M) = \mathbb{C}G \otimes_{\mathbb{C}U} \mathbb{C}U e_M = \mathbb{C}G \otimes_{\mathbb{C}U} e_M \cong \mathbb{C}G e_M$$

and the map

$$\begin{array}{rcl} \theta_M : e_M \mathbb{C}G e_M & \xrightarrow{\sim} & \mathcal{H}(G, U, M) \\ e_M g e_M & \mapsto & \phi_g \end{array} \quad \text{where} \quad \begin{array}{rcl} \phi_g : \mathbb{C}G e_M & \longrightarrow & \mathbb{C}G e_M \\ k e_M & \mapsto & k e_M g e_M, \end{array} \tag{2.2}$$

is an algebra anti-isomorphism (i.e. $\theta_M(ab) = \theta_M(b)\theta_M(a)$).

The following theorem connects the representation theory of $G$ to the representation theory of $\mathcal{H}(G, U, M)$. Theorem 2.3 and the Corollary are in [CR81, Theorem 11.25].

**Theorem 2.3 (Double Centralizer Theorem).** *Let $V$ be a $G$-module and let $\mathcal{H} = \mathrm{End}_{\mathbb{C}G}(V)$. Then, as a $G$-module,*

$$V \cong \bigoplus_{\gamma \in \hat{G}} \dim(\mathcal{H}^\gamma) G^\gamma$$

*if and only if, as an $\mathcal{H}$-module,*

$$V \cong \bigoplus_{\gamma \in \hat{G}} \dim(G^\gamma) \mathcal{H}^\gamma,$$

*where $\{\mathcal{H}^\gamma \mid \gamma \in \hat{G}, \mathcal{H}^\gamma \neq 0\}$ is the set of irreducible $\mathcal{H}$-modules.*

**Corollary 2.4.** *Suppose $U$ is a subgroup of $G$. Let $M \cong \mathbb{C}Ue_M$ be an irreducible $U$-module. Write $\mathcal{H} = \mathcal{H}(G, U, M)$. Then the map*

$$\begin{array}{ccc} \{G\text{-submodules of } \mathbb{C}Ge_M\} & \longrightarrow & \{\mathcal{H}\text{-modules}\} \\ V & \mapsto & e_M V \end{array}$$

*is a bijection that sends $G^\gamma \mapsto \mathcal{H}^\gamma$ for every irreducible $G^\gamma$ that is isomorphic to a submodule of $\mathrm{Ind}_U^G(M)$ ($G^\gamma \hookrightarrow \mathrm{Ind}_U^G(M)$).*

## 2.2 Chevalley Groups

There are two common approaches used to define finite Chevalley groups. One strategy considers the subgroup of elements in an algebraic group fixed under a Frobenius map (see [Car85]). The approach here, however, begins with a pair $(\mathfrak{g}, V)$ of a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module $V$, and constructs the Chevalley group from a variant of the exponential map. This perspective gives an explicit construction of the elements, which will prove useful in the computations that follow (see also [Ste67]).

### 2.2.1 Lie algebra set-up

A *finite dimensional Lie algebra* $\mathfrak{g}$ is a finite dimensional vector space over $\mathbb{C}$ with a $\mathbb{C}$-bilinear map

$$\begin{array}{rcl} [\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ (X, Y) & \mapsto & [X, Y] \end{array}$$

such that

(a) $[X, X] = 0$, for all $X \in \mathfrak{g}$,

(b) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, for $X, Y, Z \in \mathfrak{g}$.

Note that if $A$ is any algebra, then the bracket $[a, b] = ab - ba$ for $a, b \in A$ gives a Lie algebra structure to $A$.

A $\mathfrak{g}$-module is a vector space $V$ with a map

$$\begin{array}{rcl} \mathfrak{g} \times V & \longrightarrow & V \\ (X, v) & \mapsto & Xv \end{array}$$

such that

$$\begin{array}{rl} (aX + bY)v = aXv + bYv, & \text{for all } a, b \in \mathbb{C},\ X, Y \in \mathfrak{g},\ v \in V, \\ X(av_1 + bv_2) = aXv_1 + bXv_2, & \text{for all } a, b \in \mathbb{C},\ X \in \mathfrak{g},\ v_1, v_2 \in V, \\ [X, Y]v = X(Yv) - Y(Xv), & \text{for all } X, Y \in \mathfrak{g},\ v \in V. \end{array}$$

A Lie algebra $\mathfrak{g}$ acts *diagonally* on $V$ if there exists a basis $\{v_1, \ldots, v_r\}$ of $V$ such that

$$Xv_i \in \mathbb{C}v_i, \qquad \text{for all } X \in \mathfrak{g} \text{ and } i = 1, 2, \ldots, r.$$

The center of $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

Let $\gamma \in \hat{\mathfrak{g}}$ index the irreducible $\mathfrak{g}$-modules $\mathfrak{g}^\gamma$. A Lie algebra $\mathfrak{g}$ is *reductive* if for every finite-dimensional module $V$ on which $Z(\mathfrak{g})$ acts diagonally,

$$V = \bigoplus_{\gamma \in \hat{\mathfrak{g}}} m_\gamma(V)\mathfrak{g}^\gamma, \qquad \text{where } m_\gamma(V) \in \mathbb{Z}_{\geq 0}.$$

If $\mathfrak{g}$ is reductive, then

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_s \qquad \text{where } \mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}] \text{ is semisimple.}$$

Let $\mathfrak{g}$ be a reductive Lie algebra. A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra satisfying

(a) $\mathfrak{h}^k = [\mathfrak{h}^{k-1}, \mathfrak{h}] = 0$ for some $k \geq 1$ ($\mathfrak{h}$ is nilpotent),

(b) $\mathfrak{h} = \{X \in \mathfrak{g} \mid [X, H] \in \mathfrak{h}, H \in \mathfrak{h}\}$ ($\mathfrak{h}$ is its own normalizer).

If $\mathfrak{h}_s$ is a Cartan subalgebra of $\mathfrak{g}_s$, then $\mathfrak{h} = Z(\mathfrak{g}) \oplus \mathfrak{h}_s$ is a Cartan subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C}) \qquad \text{and} \qquad \mathfrak{h}_s^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_s, \mathbb{C}).$$

As an $\mathfrak{h}_s$-module, $\mathfrak{g}_s$ decomposes

$$\mathfrak{g}_s \cong \mathfrak{h}_s \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{h}_s^*} (\mathfrak{g}_s)_\alpha, \quad \text{where} \quad (\mathfrak{g}_s)_\alpha = \langle X \in \mathfrak{g}_s \mid [H, X] = \alpha(H)X, H \in \mathfrak{h}_s \rangle.$$

The set of *roots* of $\mathfrak{g}_s$ is $R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, (\mathfrak{g}_s)_\alpha \neq 0\}$. A set of *simple roots* is a subset $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\} \subseteq R$ of the roots such that

(a) $\mathbb{Q} \otimes_{\mathbb{C}} \mathfrak{h}_s^* = \mathbb{Q}\text{-span}\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ with $\{\alpha_1, \ldots, \alpha_\ell\}$ linearly independent,

(b) Every root $\beta \in R$ can be written as $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_\ell\alpha_\ell$ with either $\{c_1, \ldots, c_\ell\} \subseteq \mathbb{Z}_{\geq 0}$ or $\{c_1, \ldots, c_\ell\} \subseteq \mathbb{Z}_{\leq 0}$.

Every choice of simple roots splits the set of roots $R$ into *positive roots*

$$R^+ = \{\beta \in R \mid \beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_\ell\alpha_\ell,\ c_i \in \mathbb{Z}_{\geq 0}\}$$

and *negative roots*

$$R^- = \{\beta \in R \mid \beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_\ell\alpha_\ell,\ c_i \in \mathbb{Z}_{\le 0}\}.$$

In fact, $R^- = -R^+$.

**Example**. Consider the Lie algebra $\mathfrak{gl}_2 = \mathfrak{gl}_2(\mathbb{C}) = \{2 \times 2 \text{ matrices with entries in } \mathbb{C}\}$, and bracket $[\cdot,\cdot]$ given by $[X, Y] = XY - YX$. Write

$$\mathfrak{gl}_2 = Z(\mathfrak{gl}_2) \oplus \mathfrak{sl}_2,$$

where $Z(\mathfrak{gl}_2) = \{\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \mid c \in \mathbb{C}\}$ and $\mathfrak{sl}_2 = \{X \in \mathfrak{gl}_2 \mid \mathrm{tr}(X) = 0\}$. Also,

$$\mathfrak{sl}_2 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \mid a \in \mathbb{C}\} \oplus \{\left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right) \mid c \in \mathbb{C}\} \oplus \{\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right) \mid c \in \mathbb{C}\}.$$

Let $\alpha \in \mathfrak{h}^* = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a, b \in \mathbb{C}\}^*$ be the simple root given by $\alpha\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) = a - b$, so that $R^+ = \{\alpha\}$ and $R^- = \{-\alpha\}$.

### 2.2.2 From a Chevalley basis of $\mathfrak{g}_s$ to a Chevalley group

For every pair of roots $\alpha, -\alpha$, there exists a Lie algebra isomorphism $\phi_\alpha : \mathfrak{sl}_2 \to \langle \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \rangle$. Under this isomorphism, let

$$X_\alpha = \phi_\alpha\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in (\mathfrak{g}_s)_\alpha, \qquad H_\alpha = \phi_\alpha\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{h}_s, \qquad X_{-\alpha} = \phi_\alpha\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in (\mathfrak{g}_s)_{-\alpha}.$$

Note that $[X_\alpha, X_{-\alpha}] = H_\alpha$. The following classical result can be found in [Hum72, Theorem 25.2].

**Theorem 2.5 (Chevalley Basis).** *There is a choice of the $\phi_\alpha$ such that the set $\{X_\alpha, H_{\alpha_i} \mid \alpha \in R, 1 \le i \le \ell\}$ is a basis of $\mathfrak{g}_s$ satisfying*

*(a) $H_\alpha$ is a $\mathbb{Z}$-linear combination of $H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_\ell}$.*

*(b) Let $\alpha, \beta \in R$ such that $\beta \neq \pm\alpha$, and suppose $l, r \in \mathbb{Z}_{\ge 0}$ are maximal such*

$$\{\beta - l\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + r\alpha\} \subseteq R.$$

*Then*

$$[X_\alpha, X_\beta] = \begin{cases} \pm(l+1)X_{\alpha+\beta}, & \text{if } r \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

A basis as in Theorem 2.5 is a *Chevalley basis* of $\mathfrak{g}_s$ (For an analysis of the choices involved see [Sam69] and [Tit66]).

Let $V$ be a finite dimensional $\mathfrak{g}$-module such that $V$ has a $\mathbb{C}$-basis $\{v_1, v_2, \ldots, v_r\}$ that satisfies

(a) There exists a $\mathbb{C}$-basis $\{H_1, \ldots, H_n\}$ of $\mathfrak{h}$ such that

(1) $H_{\alpha_i} \in \mathbb{Z}_{\geq 0}$-span$\{H_1, \ldots, H_n\}$,

(2) $H_i v_j \in \mathbb{Z} v_j$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$.

(3) $\dim_{\mathbb{Z}}(\mathbb{Z}\text{-span}\{H_1, H_2, \ldots, H_n\}) \leq \dim_{\mathbb{C}}(\mathfrak{h})$.

(b) $\dfrac{X_\alpha^n}{n!} v_i \in \mathbb{Z}$-span$\{v_1, v_2, \ldots, v_r\}$ for $\alpha \in R$, $n \in \mathbb{Z}_{\geq 0}$ and $i = 1, 2, \ldots, r$.

(c) $\dim_{\mathbb{Z}}(\mathbb{Z}\text{-span}\{v_1, v_2, \ldots, v_r\}) \leq \dim_{\mathbb{C}}(V)$.

(Condition (a) guarantees that $Z(\mathfrak{g})$ acts diagonally. If $Z(\mathfrak{g}) = 0$, then the existence of such a basis is guaranteed by a theorem of Kostant [Hum72, Theorem 27.1]).

Let $V$ be a finite dimensional $\mathfrak{g}$-module that satisfies (a)-(c) above. For $H \in \mathfrak{h}$, let

$$\binom{H}{n} = \frac{(H-1)(H-2)\cdots(H-(n+1))}{n!} \in \mathrm{End}(V).$$

As a transformation of the basis $v_1, v_2, \ldots, v_r$, the element $H_i = \mathrm{diag}(h_1, h_2, \ldots, h_r) \in \mathrm{End}(V)$ with $h_j \in \mathbb{Z}$. Let $y \in \mathbb{C}^*$. Temporarily abandon precision (i.e. allow infinite sums) to obtain

$$\begin{aligned} \sum_{n\geq 0} (y-1)^n \binom{H_i}{n} &= \mathrm{diag}\Big(\sum_{n\geq 0}(y-1)^n \tbinom{h_1}{n}, \sum_{n\geq 0}(y-1)^n \tbinom{h_2}{n}, \ldots, \sum_{n\geq 0}(y-1)^n \tbinom{h_r}{n}\Big) \\ &= \mathrm{diag}(y^{h_1}, y^{h_2}, \ldots, y^{h_r}) \in \mathrm{End}(V) \quad \text{(by the binomial theorem).} \end{aligned}$$

This computation motivates some of the definitions below.

Let

$$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{H_1, H_2, \ldots, H_n\}. \tag{2.3}$$

The finite field $\mathbb{F}_q$ with $q$ elements has a multiplicative group $\mathbb{F}_q^*$ and an additive group $\mathbb{F}_q^+$. Let

$$V_q = \mathbb{F}_q\text{-span}\{v_1, v_2, \ldots, v_r\}. \tag{2.4}$$

The *finite reductive Chevalley group* $G_V \subseteq GL(V_q)$ is

$$G_V = \langle x_\alpha(a), h_H(b) \mid \alpha \in R, H \in \mathfrak{h}_{\mathbb{Z}}, a \in \mathbb{F}_q, b \in \mathbb{F}_q^* \rangle,$$

where

$$x_\alpha(a) = \sum_{n\geq 0} a^n \frac{X_\alpha^n}{n!}, \quad \text{and} \tag{2.5}$$

$$h_H(b) = \mathrm{diag}(b^{\lambda_1(H)}, b^{\lambda_2(H)}, \ldots, b^{\lambda_r(H)}), \quad \text{where } Hv_i = \lambda_i(H)v_i. \tag{2.6}$$

**Remarks.**

1. If we "allow" infinite sums, then

$$h_{H_i}(b) = \sum_{n\geq 0} (b-1)^n \binom{H_i}{n}.$$

2. If $\mathfrak{g} = \mathfrak{g}_s$, then $G_V = \langle x_\alpha(t) \rangle$.

**Example**. Suppose (as before) $\mathfrak{g} = \mathfrak{gl}_2$ and let

$$V = \mathbb{C}\text{-span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

be the natural $\mathfrak{g}$-module given by matrix multiplication. Then $\mathfrak{h}$ has a basis

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C} \right\} = \mathbb{C}\text{-span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

By direct computation,

$$x_\alpha(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h_{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)}(t) = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} \quad \text{for } a, b \in \mathbb{Z},$$

and $G_V = GL_2(\mathbb{F}_q)$ (the general linear group).

## 2.3 Some combinatorics of the symmetric group

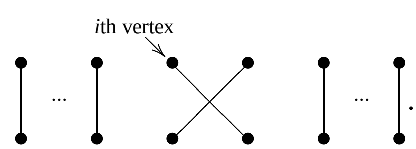
This section describes some of the pertinent combinatorial objects and techniques.

### 2.3.1 A pictorial version of the symmetric group $S_n$

Let $s_i \in S_n$ be the simple reflection that switches $i$ and $i+1$, and fixes everything else. The group $S_n$ can be presented by generators $s_1, s_2, \ldots, s_{n-1}$ and relations

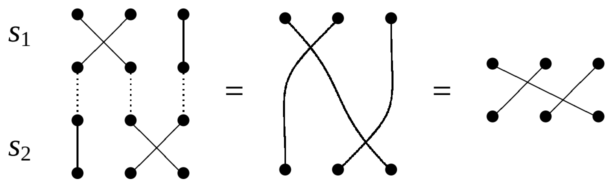
$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \quad \text{for } |i-j| > 1.$$

Using two rows of $n$ vertices and strands between the top and bottom vertices, we may pictorially describe permutations in the following way. View

$s_i$ as 

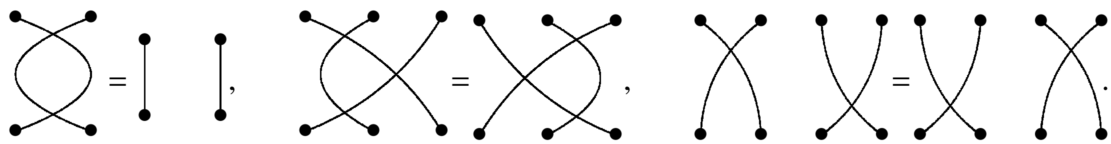
.

Multiplication in $S_n$ corresponds to concatenation of diagrams, so for example,




$= s_2 s_1.$

Therefore, $S_n$ is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations

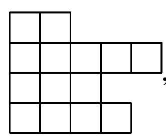


### 2.3.2 Compositions, partitions, and tableaux

A *composition* $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is a sequence of positive integers. The *size* of $\mu$ is $|\mu| = \mu_1 + \mu_2 + \cdots + \mu_r$, the *length* of $\mu$ is $\ell(\mu) = r$ and

$$\mu_{\leq} = \{\mu_{\leq 1}, \mu_{\leq 2}, \ldots, \mu_{\leq r}\}, \quad \text{where } \mu_{\leq i} = \mu_1 + \mu_2 + \cdots + \mu_i. \tag{2.7}$$

If $|\mu| = n$, then $\mu$ is *a composition of n* and we write $\mu \models n$. View $\mu$ as a collection of boxes aligned to the left. If a box $x$ is in the $i$th row and $j$th column of $\mu$, then the *content* of $x$ is $c(x) = i - j$. For example, if

$\mu = (2, 5, 3, 4) =$  ,

then $|\mu| = 14$, $\ell(\mu) = 4$, $\mu_{\leq} = (2, 7, 10, 14)$, and the contents of the boxes are

| | | | | |
|---|---|---|---|---|
| 0 | 1 | | | |
| -1 | 0 | 1 | 2 | 3 |
| -2 | -1 | 0 | | |
| -3 | -2 | -1 | 0 | |

.

Alternatively, $\mu_{\leq}$ coincides with the numbers in the boxes at the end of the rows in the diagram

| | | | | |
|---|---|---|---|---|
| 1 | 2 | | | |
| 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | | |
| 11 | 12 | 13 | 14 | |

.

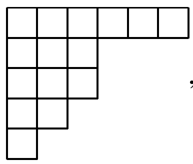
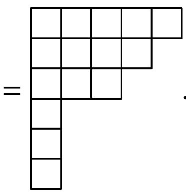
A *partition* $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ is a composition where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r > 0$. If $|\nu| = n$, then $\nu$ is *a partition of n* and we write $\nu \vdash n$. Let

$$\mathcal{P} = \{\text{partitions}\} \qquad \text{and} \qquad \mathcal{P}_n = \{\nu \vdash n\}. \tag{2.8}$$

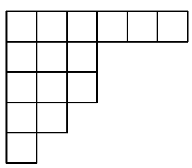
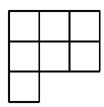
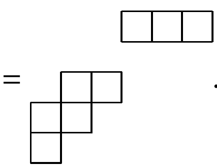
Suppose $\nu \in \mathcal{P}$. The *conjugate partition* $\nu' = (\nu'_1, \nu'_2, \dots, \nu'_\ell)$ is given by

$$\nu'_i = \mathrm{Card}\{j \mid \nu_j \geq i\}.$$

In terms of diagrams, $\nu'$ is the collection of boxes obtained by flipping $\nu$ across its main diagonal. For example,

if $\nu =$  , then $\nu' =$  .

Suppose $\nu, \mu$ are partitions. If $\nu_i \geq \mu_i$ for all $1 \leq i \leq \ell(\mu)$, then the *skew partition* $\nu/\mu$ is the collection of boxes obtained by removing the boxes in $\mu$ from the upper left-hand corner of the diagram $\lambda$. For example, if

$\nu =$  and $\mu =$  , then $\nu/\mu =$  .

A *horizontal strip* $\nu/\mu$ is a skew shape such that no column contains more than one box. Note that if $\mu = \emptyset$, then $\nu/\mu$ is a partition.

A *column strict tableau Q of shape* $\nu/\mu$ is a filling of the boxes of $\nu/\mu$ by positive integers such that

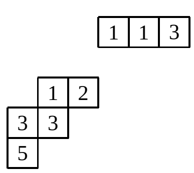
(a) the entries strictly increase along columns,

(b) the entries weakly increase along rows.

The *weight of* $Q$ is the composition $\mathrm{wt}(Q) = (\mathrm{wt}(Q)_1, \mathrm{wt}(Q)_2, \dots)$ given by

$$\mathrm{wt}(Q)_i = \text{number of } i \text{ in } Q.$$

For example,

$Q =$ 

has $\mathrm{wt}(Q) = (3, 1, 3, 0, 1)$.

### 2.3.3 Symmetric functions

The symmetric group $S_n$ acts on the set of variables $\{x_1, x_2, \ldots, x_n\}$ by permuting the indices. The *ring of symmetric polynomials in the variables* $\{x_1, x_2, \ldots, x_n\}$ is

$$\Lambda_n(x) = \{f \in \mathbb{Z}[x_1, x_2, \ldots, x_n] \mid w(f) = f, w \in S_n\}.$$

For $r \in \mathbb{Z}_{\geq 0}$, the *rth elementary symmetric polynomial* is

$$e_r(x:n) = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad \text{where by convention } e_r(x:n) = 0, \text{ if } r > n,$$

and the *rth power sum symmetric polynomial* is

$$p_r(x:n) = x_1^r + x_2^r + \cdots + x_n^r.$$

For any partition $\nu \in \mathcal{P}$, let

$$\begin{aligned} e_\nu(x:n) &= e_{\nu_1}(x:n)e_{\nu_2}(x:n)\cdots e_{\nu_\ell}(x:n) \\ p_\nu(x:n) &= p_{\nu_1}(x:n)p_{\nu_2}(x:n)\cdots p_{\nu_\ell}(x:n). \end{aligned}$$

The *Schur polynomial corresponding to* $\nu$ is

$$s_\nu(x:n) = \det(e_{\nu'_i-i+j}(x:n)), \tag{2.9}$$

for which Pieri's rule gives

$$s_\nu(x:n)s_{(n)}(x:n) = \sum_{\substack{\text{horizontal strip } \gamma/\nu \\ |\gamma/\nu|=n}} s_\gamma(x:n) \qquad \text{[Mac95, I.5.16]}. \tag{2.10}$$

For each $t \in \mathbb{C}$, the *Hall-Littlewood symmetric function* is

$$\begin{aligned} P_\nu(x:n;t) &= \sum_{w\nu\in S_n\nu} w\left(x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n} \prod_{\nu_i>\nu_j} \frac{x_i - tx_j}{x_i - x_j}\right) \\ &= \frac{1}{v_\nu(t)} \sum_{w\in S_n} w\left(x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\right), \end{aligned}$$

where $v_\nu(t) \in \mathbb{C}$ is a constant as in [Mac95, III.2]; and they satisfy

$$P_\nu(x:n;0) = s_\nu(x:n), \qquad P_{(1^n)}(x:n;t) = e_n(x:n).$$

For $t \in \mathbb{C}$,

$$\Lambda_n(x) = \mathbb{Z}\text{-span}\{e_\nu(x:n)\} = \mathbb{Z}\text{-span}\{s_\nu(x:n)\} = \mathbb{Z}\text{-span}\{P_\nu(x:n;t)\}, \tag{2.11}$$

and if we extend coefficients to $\mathbb{Q}$, then

$$\mathbb{Q} \otimes \Lambda_n(x) = \mathbb{Q}\text{-span}\{p_\nu(x:n)\}.$$

Note that for $n > m$ there is a map

$$\begin{array}{ccc} \Lambda_n(x) & \longrightarrow & \Lambda_m(x) \\ f(x_1, x_2, \ldots, x_n) & \mapsto & f(x_1, \ldots, x_m, 0, \ldots, 0) \end{array}$$

which sends $s_\nu(x : n) \mapsto s_\nu(x : m)$, $e_\nu(x : n) \mapsto e_\nu(x : m)$, etc. Let $\{x_1, x_2, \ldots\}$ be an infinite set of variables. The *Schur function* $s_\nu(x)$ is the infinite sequence

$$s_\nu(x) = (s_\nu(x:1), s_\nu(x:2), s_\nu(x:3), \ldots),$$

and one can define elementary symmetric functions $e_r(x)$, power sum symmetric functions $p_r(x)$, and Hall-Littlewood symmetric functions $P_\nu(x;t)$, analogously. The *ring of symmetric functions in the variables* $\{x_1, x_2, \ldots\}$ is

$$\Lambda(x) = \mathbb{Z}\text{-span}\{s_\nu(x) \mid \nu \in \mathcal{P}\},$$

and let

$$\Lambda_{\mathbb{C}}(x) = \mathbb{C}\text{-span}\{s_\nu(x) \mid \nu \in \mathcal{P}\}.$$

### 2.3.4 RSK correspondence

The classical RSK correspondence provides a combinatorial proof of the identity

$$\prod_{i,j>0} \frac{1}{1 - x_i y_j} = \sum_{\substack{\nu \vdash n \\ n \geq 0}} s_\nu(x) s_\nu(y) \qquad \text{[Knu70]}$$

by constructing for each $\ell \geq 0$ a bijection between the matrices $b \in M_\ell(\mathbb{Z}_{\geq 0})$ and the set of pairs $(P(b), Q(b))$ of column strict tableaux with the same shape. The bijection is as follows.

If $P$ is a column strict tableau and $j \in \mathbb{Z}_{>0}$, let $P \leftarrow j$ be the column strict tableau given by the following algorithm

(a) Insert $j$ into the the first column of $P$ by displacing the smallest number $\geq j$. If all numbers are $< j$, then place $j$ at the bottom of the first column.

(b) Iterate this insertion by inserting the displaced entry into the next column.

(c) Stop when the insertion does not displace an entry.

A *two-line array* $\begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ is a two-rowed array with $i_1 \leq i_2 \leq \cdots \leq i_n$ and $j_k \geq j_{k+1}$ if $i_k = i_{k+1}$. If $b \in M_\ell(\mathbb{Z}_{\geq 0})$, then let $\vec{b}$ be the two-line array with $b_{ij}$ pairs $\binom{i}{j}$.

For $b \in M_\ell(\mathbb{Z}_{\geq 0})$, suppose

$$\vec{b} = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}.$$

Then the pair $(P(b), Q(b))$ is the final pair in the sequence

$$(\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n) = (P(b), Q(b)),$$

where $(P_k, Q_k)$ is a pair of column strict tableaux with the same shape given by

$P_k = P_{k-1} \leftarrow j_k$ and $Q_k$ is defined by $\mathrm{sh}(Q_k) = \mathrm{sh}(P_k)$ with $i_k$ in the new box $\mathrm{sh}(Q_k)/\mathrm{sh}(Q_{k-1})$.

For example,

$$b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{corresponds to} \quad \vec{b} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 3 & 2 \end{pmatrix}$$

and provides the sequence

$$(\emptyset, \emptyset), \left(\boxed{2}, \boxed{1}\right), \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right), \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}\right), \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\right)$$

so that

$$(P(b), Q(b)) = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\right).$$

# Chapter 3

# Unipotent Hecke algebras

## 3.1 The algebras

### 3.1.1 Important subgroups of a Chevalley group

Let $G = G_V$ be a Chevalley group defined with a $\mathfrak{g}$-module $V$ as in Section 2.2.2. Recall that $R^+$ is the set of positive roots according to some fixed set of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ of $\mathfrak{g}$ (Section 2.2.1). The group $G$ contains a subgroup $U$ given by

$$U = \langle x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q \rangle,$$

which decomposes as

$$U = \prod_{\alpha \in R^+} U_\alpha, \qquad \text{where} \quad U_\alpha = \langle x_\alpha(t) \mid t \in \mathbb{F}_q \rangle,$$

with uniqueness of expression for any fixed ordering of the positive roots [Ste67, Lemma 18]. For each $\alpha \in R^+$, the map

$$\begin{array}{rcl} U_\alpha & \xrightarrow{\sim} & \mathbb{F}_q^+ \\ x_\alpha(t) & \mapsto & t \end{array}$$

is a group homomorphism.

For $\alpha \in R$, define the maps

$$\begin{array}{rrcl} s_\alpha : & \mathfrak{h}^* & \longrightarrow & \mathfrak{h}^* \\ & \gamma & \mapsto & \gamma - \gamma(H_\alpha)\alpha, \end{array} \tag{3.1}$$

$$\begin{array}{rrcl} s_\alpha : & \mathfrak{h} = Z(\mathfrak{g}) \oplus \mathfrak{h}_s & \longrightarrow & \mathfrak{h} \\ & H + H_\beta & \mapsto & H + H_\beta - \beta(H_\alpha)H_\alpha, \quad \text{for } \beta \in R. \end{array} \tag{3.2}$$

The *Weyl group of G* is $W = \langle s_\alpha \mid \alpha \in R \rangle$ and has a presentation given by

$$W = \langle s_1, s_2, \ldots, s_\ell \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1, 1 \le i \ne j \le \ell \rangle, \quad m_{ij} \in \mathbb{Z}_{>0}, \; s_i = s_{\alpha_i}.$$

If $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ with $r$ minimal, then the *length* of $w$ is $\ell(w) = r$.

Let $\mathfrak{h}_\mathbb{Z}$ be as in (2.3). If $q > 3$, then the subgroup

$$T = \langle h_H(t) \mid H \in \mathfrak{h}_\mathbb{Z}, t \in \mathbb{F}_q^* \rangle$$

has its normalizer in $G$ given by

$$N = \langle w_\alpha(t), h \mid \alpha \in R, h \in T, t \in \mathbb{F}_q^* \rangle, \quad \text{where } w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t).$$

If $\alpha \in R$, then $h_{H_\alpha}(t) = w_\alpha(t)w_\alpha(1)^{-1}$. Write $h_\alpha(t) = h_{H_\alpha}(t)$ and $h_i(t) = h_{\alpha_i}(t)$.

There is a natural surjection from $N$ onto the Weyl group $W$ with kernel $T$ given by

$$\begin{array}{rccll} \pi: & N & \longrightarrow & W & \\ & w_\alpha(t) & \mapsto & s_\alpha, & \text{for } \alpha \in R, t \in \mathbb{F}_q^*, \\ & h & \mapsto & 1, & \text{for } h \in T. \end{array} \tag{3.3}$$

Suppose $v \in N$. Then for each minimal expression

$$\pi(v) = s_{i_1}s_{i_2}\dots s_{i_r}, \quad \text{with } \ell(\pi(v)) = r,$$

there is a unique decomposition of $v$ as

$$v = v_1v_2\cdots v_rv_T, \quad \text{where } v_k = w_{i_k}(1) \text{ and } v_T \in T. \tag{3.4}$$

Write

$$\xi_i = w_i(1). \tag{3.5}$$

### 3.1.2 Unipotent Hecke algebras

Let $G$ be a finite Chevalley group. Fix a nontrivial homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$. If

$$\begin{array}{rccc} \mu: & R^+ & \to & \mathbb{F}_q \\ & \alpha & \mapsto & \mu_\alpha \end{array} \qquad \text{satisfies } \mu_\alpha = 0 \text{ for all } \alpha \text{ not simple}, \tag{3.6}$$

then the map

$$\begin{array}{rccc} \psi_\mu: & U & \longrightarrow & \mathbb{C}^* \\ & x_\alpha(t) & \mapsto & \psi(\mu_\alpha t) \end{array} \tag{3.7}$$

is a linear character of $U$. In fact, with the exception of a few degenerate special cases of $G$ (which can be avoided if $q > 3$), all linear characters of $U$ are of this form [Yok69a, Theorem 1].

The *unipotent Hecke algebra* $\mathcal{H}(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\psi_\mu)). \tag{3.8}$$

Using the anti-isomorphism (2.2), view $\mathcal{H}_\mu$ as the subset of $\mathbb{C}G$

$$\mathcal{H}_\mu = e_\mu\mathbb{C}Ge_\mu, \qquad \text{where} \quad e_\mu = \frac{1}{|U|}\sum_{u\in U}\psi_\mu(u^{-1})u. \tag{3.9}$$

### 3.1.3 The choices $\mu$ and $\psi$

This section assumes all linear characters of $U$ are of the form $\psi_\mu$ for some nontrivial homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$ and $\mu$ as in (3.6).

The group $T$ normalizes $U_\alpha$, since

$$hx_\alpha(t)h^{-1} = x_\alpha(\alpha(h)t), \qquad \text{for } \alpha \in R^+,\ t \in \mathbb{F}_q,\ h \in T,\ \alpha(h) \in \mathbb{F}_q^*.$$

Thus, if $\chi : U \to \mathbb{C}^*$ is a linear character, then

$$\begin{array}{rccc} {}^h\chi : & U & \longrightarrow & \mathbb{C}^* \\ & x_\alpha(t) & \mapsto & \chi(hx_\alpha(t)h^{-1}) \end{array}$$

is a linear character for every $h \in T$.

**Proposition 3.1.** *Suppose $\chi : U \to \mathbb{C}^*$ is a linear character. For every $h \in T$,*

$$\mathrm{Ind}_U^G(\chi) = \mathrm{Ind}_U^G({}^h\chi).$$

*Proof.* Recall that

$$\mathrm{Ind}_U^G({}^h\psi_\mu)(g) = \frac{1}{|U|} \sum_{\substack{x \in G \\ xgx^{-1} \in U}} {}^h\psi_\mu(xgx^{-1}) = \frac{1}{|U|} \sum_{\substack{x \in G \\ xgx^{-1} \in U}} \psi_\mu(hxgx^{-1}h^{-1}).$$

Since $T$ normalizes $U$, the sum over all $x \in G$ such that $xgx^{-1} \in U$ is the same as the sum over $hx \in G$ such that $hxgx^{-1}h^{-1} \in U$. Thus,

$$\mathrm{Ind}_U^G({}^h\psi_\mu)(g) = \frac{1}{|U|} \sum_{\substack{x' \in G \\ x'gx'^{-1} \in U}} \psi_\mu(x'gx'^{-1}) = \mathrm{Ind}_U^G(\psi_\mu)(g). \qquad \square$$

The *type* of a linear character $\chi : U \to \mathbb{C}^*$ is the set $J \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ such that

$$\chi(x_{\alpha_i}(t)) \neq 1 \text{ for all } t \in \mathbb{F}_q \qquad \text{if and only if} \qquad \alpha_i \in J.$$

A unipotent Hecke algebra $\mathcal{H}(G, U, \chi)$ has *type* $J$ if $\chi$ has type $J$.

**Proposition 3.2.**

(a) *Fix a nontrivial homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$. If $\psi' : \mathbb{F}_q^+ \to \mathbb{C}^*$ is a homomorphism, then there exists $k \in \mathbb{F}_q$ such that*

$$\psi'(t) = \psi(kt), \qquad \textit{for all } t \in \mathbb{F}_q^+.$$

(b) *The linear characters $\chi : U \to \mathbb{C}^*$ and ${}^h\chi : U \to \mathbb{C}^*$ have the same type for any $h \in T$.*

(c) *Let $J \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ and $\mathcal{S}_J = \{$type $J$ linear characters of $U\}$. Then the number of $T$-orbits of $\mathcal{S}_J$ is*

$$\frac{|Z_J|(q-1)^{|J|}}{|T|}, \qquad \text{where } Z_J = \{h \in T \mid \alpha(h) = 1, \text{ for } \alpha \in J\}.$$

(d) *The number of distinct (nonisomorphic) type $J$ unipotent Hecke algebras is at most*

$$\frac{|Z_J|(q-1)^{|J|}}{|T|}.$$

*Proof.* (a) The map

$$\begin{array}{ccc} \mathbb{F}_q^+ & \longrightarrow & \mathrm{Hom}(\mathbb{F}_q^+, \mathbb{C}^*) \\ k & \longmapsto & \begin{array}{rcl} \psi_k : \mathbb{F}_q^+ & \to & \mathbb{C}^* \\ t & \mapsto & \psi(kt) \end{array} \end{array}$$

is a group homomorphism, and the kernel is trivial because $\psi$ is nontrivial. Since $|\mathrm{Hom}(\mathbb{F}_q^+, \mathbb{C})| = q$, the map is also surjective.

(b) Since $U_\alpha \cong \mathbb{F}_q^+$, the restriction of $\chi$ to $U_\alpha$ gives a linear character of $\mathbb{F}_q^+$. Part (a) implies that for every $\alpha \in R^+$ there exists $k_\alpha \in \mathbb{F}_q$ such that

$$\chi(x_\alpha(t)) = \psi(k_\alpha t)$$

Furthermore, if $\chi$ has type $J$, then $k_\alpha \neq 0$ if and only if $\alpha \in J$. Write

$$\chi = \psi_{(k_1, \ldots, k_\ell)} \qquad \text{where } k_i = k_{\alpha_i}.$$

Since $hx_\alpha(t)h^{-1} = x_\alpha(\alpha(h)t)$ for $h \in T$,

$${}^h\chi = \psi_{(\alpha_1(h)k_1, \ldots, \alpha_\ell(h)k_\ell)},$$

and $\alpha(h)k_\alpha \neq 0$ if and only if $k_\alpha \neq 0$ if and only if $\alpha \in J$. Thus, the action of $T$ preserves the type of $\chi$.

(c) Suppose a group $K$ acts on a set $\mathcal{S}$. Then by [Isa94, Theorem 4.18] the number of orbits of the $K$ action is

$$\frac{1}{|K|} \sum_{g \in K} \mathrm{Card}\{s \in \mathcal{S} \mid g(s) = s\}.$$

In this case, $K = T$ acts on the set

$$\mathcal{S} = \mathcal{S}_J = \{\psi_{(k_1, k_2, \ldots, k_\ell)} \mid k_i = 0, \text{ unless } \alpha_i \in J\}.$$

Furthermore,

$${}^h\psi_{(k_1, k_2, \ldots, k_\ell)} = \psi_{(k_1, k_2, \ldots, k_\ell)} \quad \text{if and only if} \quad \alpha_i(h) = 1 \text{ for all } \alpha_i \in J.$$

Finally, $|\mathcal{S}_J| = (q-1)^{|J|}$. □

**Examples.**

1. If $J = \emptyset$, then $|J| = 0$ and $|Z_J| = |T|$, so there is a unique unipotent Hecke algebra of type $J$; in this case, $\psi_\mu$ is trivial and $\mathcal{H}_\mu$ is the Yokonuma algebra.

2. A character is *in general position* if it has type $J = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. In this case, $Z_J$ is the center of $G$, and $|J| = \ell$, so the number of type $J$ unipotent Hecke algebras is at most
$$\frac{|Z(G)|(q-1)^\ell}{|T|}.$$

### 3.1.4 A natural basis

The group $G$ has a double-coset decomposition

$$G = \bigsqcup_{v \in N} UvU, \qquad \text{[Ste67, Theorem 4 and } B = UT] \tag{3.10}$$

and if

$$\begin{aligned} N_\mu &= \{v \in N \mid e_\mu v e_\mu \neq 0\} \\ &= \{v \in N \mid u, vuv^{-1} \in U \text{ implies } \psi_\mu(u) = \psi_\mu(vuv^{-1})\} \end{aligned} \tag{3.11}$$

then the set $\{e_\mu v e_\mu \mid v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$ [CR81, Prop. 11.30].

Theorem 3.7 gives a set of relations similar to those of the Yokonuma algebra (Example 1, below) for evaluating the product

$$(e_\mu u e_\mu)(e_\mu v e_\mu), \qquad \text{with } u, v \in N_\mu.$$

in any unipotent Hecke algebra $\mathcal{H}_\mu$.

**Examples.**

**1. The Yokonuma Hecke algebra.** If $\mu_\alpha = 0$ for all positive roots $\alpha$, then $\psi_\mu = \mathbb{1}$ is the trivial character and $N_{\mathbf{1}} = N$. Let $T_v = e_{\mathbf{1}} v e_{\mathbf{1}}$ for $v \in N$, with $T_i = T_{\xi_i}$ ($\xi_i$ as in (3.5)) and $T_H(t) = T_{h_H(t)}$. If $v \in N$ has a decomposition $v = v_1 v_2 \cdots v_r v_T$ (as in (3.4)), then

$$T_v = T_{i_1} T_{i_2} \cdots T_{i_r} T_{v_T} \qquad \text{(See Chapter 6).}$$

Thus, the Yokonuma algebra $\mathcal{H}_{\mathbf{1}}$ has generators $T_i$, $T_h$ for $1 \leq i \leq \ell$, $h \in T$ (see [Yok69b]) with relations,

$$\begin{aligned}
T_i^2 &= q^{-1} T_{H_{\alpha_i}}(-1) + q^{-1} \sum_{t \in \mathbb{F}_q^*} T_{H_{\alpha_i}}(t^{-1}) T_i, && 1 \leq i \leq \ell, \\
\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ terms}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ terms}}, && (s_i s_j)^{m_{ij}} = 1, \\
T_i T_h &= T_{s_i h} T_i, && h \in T, \\
T_h T_k &= T_{hk}, && h, k \in T.
\end{aligned}$$

These relations give an "efficient" way to compute arbitrary products $(e_{\mathbf{1}}ue_{\mathbf{1}})(e_{\mathbf{1}}ve_{\mathbf{1}})$ in $\mathcal{H}_{\mathbf{1}}$. There is a surjective map from the Yokonuma algebra onto the Iwahori-Hecke algebra that sends $T_h \mapsto 1$ for all $h \in T$. "Setting $T_h = 1$" in the Yokonuma algebra relations recovers relations for the Iwahori-Hecke algebra,

$$T_i^2 = q^{-1} + q^{-1}(q-1)T_i, \qquad \underbrace{T_iT_j\cdots}_{m_{ij}\text{ terms}} = \underbrace{T_jT_i\cdots}_{m_{ij}\text{ terms}}.$$

Furthermore, there is a surjective map from the Iwahori Hecke algebra onto the Weyl group given by mapping $T_i \mapsto s_i$ and $q \mapsto 1$. Thus, by "setting $T_i = s_i$ and $q = 1$" we retrieve the Coxeter relations of $W$,

$$s_i^2 = 1, \qquad \underbrace{s_is_js_i\cdots}_{m_{ij}\text{ terms}} = \underbrace{s_js_is_j\cdots}_{m_{ij}\text{ terms}}.$$

For a more detailed discussion of the Yokonuma algebra, see Chapter 6.

**2. The Gelfand-Graev Hecke algebra.** By definition, if $\mu_\alpha \neq 0$ for all simple roots $\alpha$, then $\psi_\mu$ is in general position. In this case, the Gelfand-Graev character $\mathrm{Ind}_U^G(\psi_\mu)$ is multiplicity free as a $G$-module ([Yok68],[Ste67, Theorem 49]). The corresponding Hecke algebra $\mathcal{H}_\mu$ is therefore commutative. However, decomposing the product $(e_\mu ue_\mu)(e_\mu ve_\mu)$ into basis elements is more challenging than in the Yokonuma case [Cha76, Cur88, Rai02].

## 3.2 Parabolic subalgebras of $\mathcal{H}_\mu$

Let $\psi_\mu : U \to G$ be as in (3.7). Fix a subset $J \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ such that

$$\text{if } \mu_{\alpha_i} \neq 0, \text{ then } \alpha_i \in J.$$

For example, if $\psi_\mu$ is in general position, then $J = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$, but if $\psi_\mu$ is trivial, then $J$ could be any subset.

Let

$$W_J = \langle s_i \in W \mid \alpha_i \in J\rangle, \quad P_J = \langle U, T, W_J\rangle \quad \text{and} \quad R_J = \mathbb{Z}\text{-span}\{\alpha_i \in J\} \cap R.$$

Then $P_J$ has subgroups

$$L_J = \langle T, W_J, U_\alpha \mid \alpha \in R_J\rangle \quad \text{and} \quad U_J = \langle U_\alpha \mid \alpha \in R^+ - R_J\rangle \tag{3.12}$$

(a Levi subgroup and the unipotent radical of $P_J$, respectively). Note that

$$U_JL_J = P_J, \quad U_J \cap L_J = 1, \quad \text{and, in fact,} \quad P_J = U_J \rtimes L_J.$$

Define the idempotents of $\mathbb{C}U$,

$$e_{\mu J} = \frac{1}{|L_J \cap U|} \sum_{u \in L_J \cap U} \psi_\mu(u^{-1})u \quad \text{and} \quad e'_J = \frac{1}{|U_J|} \sum_{u \in U_J} u, \tag{3.13}$$

so that $e_\mu = e_{\mu J} e'_J$.

The group homomorphisms

$$\begin{array}{ccc} P_J & \longrightarrow & L_J \\ lu & \mapsto & l \end{array} \quad \text{and} \quad \begin{array}{ccc} P_J & \longrightarrow & G \\ lu & \mapsto & lu \end{array} \quad \text{for } l \in L_J, u \in U_J,$$

induce maps

$$\begin{array}{rccc} \mathrm{Inf}_{L_J}^{P_J}: & \{L_J\text{-modules}\} & \longrightarrow & \{P_J\text{-modules}\} \\ & M & \mapsto & e'_J M \\ \mathrm{Ind}_{P_J}^{G}: & \{P_J\text{-modules}\} & \longrightarrow & \{G\text{-modules}\} \\ & M' & \mapsto & \mathbb{C}G \otimes_{\mathbb{C}P_J} M' \end{array},$$

whose composition is the map $\mathrm{Indf}_{L_J}^G$. Note that in the special case when $\mathbb{C}L_J e$ is an irreducible $L_J$-module with corresponding idempotent $e$, then

$$\begin{array}{rccc} \mathrm{Indf}_{L_J}^G: & \{L_J\text{-modules}\} & \longrightarrow & \{G\text{-modules}\} \\ & \mathbb{C}L_J e & \mapsto & \mathbb{C}Gee'_J. \end{array}$$

The map $\psi_\mu : U \to \mathbb{C}^*$ restricts to a linear character $\mathrm{Res}_{U \cap L_J}^U(\psi_\mu) : L_J \cap U \to \mathbb{C}^*$. To make the notation less heavy-handed, write $\psi_\mu : L_J \cap U \to \mathbb{C}^*$, for $\mathrm{Res}_{U \cap L_J}^U(\psi_\mu)$.

**Lemma 3.3.** *Let $\psi_\mu$ be as in (3.7). Then*

$$\mathrm{Ind}_U^G(\psi_\mu) \cong \mathrm{Indf}_{L_J}^G(\mathrm{Ind}_{U \cap L_J}^{L_J}(\psi_\mu)).$$

*Proof.* Recall $\mathrm{Ind}_U^G(\psi_\mu) \cong \mathbb{C}Ge_\mu$. On the other hand,

$$\mathrm{Ind}_{U \cap L_J}^{L_J}(\psi_\mu) \cong \mathbb{C}L_J e_{\mu J} \quad \text{implies} \quad \mathrm{Indf}_{L_J}^G(\mathrm{Ind}_{U \cap L_J}^{L_J}(\psi_\mu)) \cong \mathbb{C}Ge_{\mu J}e'_J,$$

where $e_{\mu J}$ is as in (3.13). But $e_{\mu J}e'_J = e_\mu$, so

$$\mathrm{Ind}_U^G(\psi_\mu) \cong \mathbb{C}Ge_\mu \cong \mathbb{C}e_{\mu J}e'_J \cong \mathrm{Indf}_{L_J}^G(\mathrm{Ind}_{U \cap L_J}^{L_J}(\psi_\mu)).$$ □

**Theorem 3.4.** *The map*

$$\begin{array}{rccc} \theta: & \mathrm{End}_{\mathbb{C}L_J}(\mathrm{Ind}_{U \cap L_J}^{L_J}(\psi_\mu)) & \longrightarrow & \mathcal{H}_\mu \\ & e_{\mu J} v e_{\mu J} & \mapsto & e_\mu v e_\mu, \quad \textit{for } v \in L_J \cap N_\mu, \end{array}$$

*is an injective algebra homomorphism.*

*Proof.* Let $v \in N \cap L_J$. Since $e_{\mu J} v e_{\mu J} \in \mathbb{C}L_J$, $e'_J \in \mathbb{C}U_J$ and $L_J \cap U_J = 1$,

$$e_{\mu J} v e_{\mu J} = 0 \quad \text{if and only if} \quad e'_J e_{\mu J} v e_{\mu J} = 0.$$

Because $L_J$ normalizes $U_J$, both $e_{\mu J}$ and $v$ commute with $e'_J$. Therefore,

$$e_{\mu J} v e_{\mu J} = 0 \quad \text{if and only if} \quad e'_J e_{\mu J} v e'_J e_{\mu J} = e_\mu v e_\mu = 0,$$

and

$$\{e_{\mu J} v e_{\mu J} \neq 0\} \quad \text{if and only if} \quad v \in N_\mu \cap L_J.$$

Consequently, $\theta$ is both well-defined and injective.

Consider

$$\theta(e_{\mu J} u e_{\mu J})\theta(e_{\mu J} v e_{\mu J}) = e_\mu u e_\mu e_\mu v e_\mu = e_{\mu J} e'_J u e_{\mu J} e'_J v e'_J e_{\mu J}.$$

Since $u$ commutes with $e'_J$,

$$\begin{aligned}\theta(e_{\mu J} u e_{\mu J})\theta(e_{\mu J} v e_{\mu J}) &= e_{\mu J} e'_J u e_{\mu J} v e'_J e_{\mu J} = e_\mu u e_{\mu J} v e_\mu \\ &= \theta(e_{\mu J} u e_{\mu J} v e_{\mu J}),\end{aligned}$$

and so $\theta$ is an algebra homomorphism. □

Write

$$\mathcal{L}_J = \theta(\mathrm{End}_{\mathbb{C}L_J}(\mathrm{Ind}_{U\cap L_J}^{L_J}(\psi_\mu))) \subseteq \mathcal{H}_\mu \tag{3.14}$$

The $\mathcal{L}_J$ are "parabolic" subalgebras of $\mathcal{H}_\mu$ in that they have a similar relationship to the representation theory of $\mathcal{H}_\mu$ as parabolic subgroups $P_J$ have with the representation theory of $G$.

### 3.2.1 Weight space decompositions for $\mathcal{H}_\mu$-modules

An important special case of Theorem 3.4 is when

$$J = J_\mu = \{\alpha_i \in \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\} \mid \mu_{\alpha_i} \neq 0\},$$

so that $\psi_\mu$ has type $J_\mu$. Write $L_\mu = L_{J_\mu}$, $W_\mu = W_{J_\mu}$, etc.

**Corollary 3.5.** *The algebra $\mathcal{L}_\mu$ is a nonzero commutative subalgebra of $\mathcal{H}_\mu$.*

*Proof.* As a character of $U \cap L_\mu$, $\psi_\mu$ is in general position, so $\mathrm{Ind}_{L_\mu\cap U}^{L_\mu}(\psi_\mu)$ is a Gelfand-Graev module and $\mathcal{L}_\mu$ is a Gelfand-Graev Hecke algebra (and therefore commutative). □

Since $\mathcal{L}_\mu$ is commutative, all the irreducible modules are one-dimensional; let $\hat{\mathcal{L}}_\mu$ be an indexing set for the irreducible modules of $\mathcal{L}_\mu$. Suppose $V$ is an $\mathcal{H}_\mu$-module. As an $\mathcal{L}_\mu$-module,

$$V \cong \bigoplus_{\gamma \in \hat{\mathcal{L}}_\mu} V_\gamma \qquad \text{where} \quad V_\gamma = \{v \in V \mid xv = \gamma(x)v, x \in \mathcal{L}_\mu\}.$$

If $\gamma \in \hat{\mathcal{L}}_\mu$, then $V_\gamma$ is the $\gamma$*-weight space* of $V$, and $V$ *has a weight* $\gamma$ if $V_\gamma \neq 0$. See Chapter 6 for an application of this decomposition.

**Examples**

1. In the Yokonuma algebra $\psi_\mu = \mathbb{1}$, $J_{\mathbf{1}} = \emptyset$ and $\mathcal{L}_{\mathbf{1}} = e_{\mathbf{1}}\mathbb{C}Te_{\mathbf{1}}$.
2. In the Gelfand-Graev Hecke algebra case, $J_\mu = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ and $\mathcal{L}_\mu = \mathcal{H}_\mu$ (confirming, but not proving, that $\mathcal{H}_\mu$ is commutative).

# 3.3 Multiplication of basis elements

This section examines the decomposition of products in terms of the natural basis

$$(e_\mu u e_\mu)(e_\mu v e_\mu) = \sum_{v' \in N_\mu} c_{uv}^{v'} (e_\mu v' e_\mu).$$

In particular, Theorem 3.7, below, gives a set of braid-like relations (similar to those of the Yokonuma algebra) for manipulating the products, and Corollary 3.9 gives a recursive formula for computing these products.

## 3.3.1 Chevalley group relations

The relations governing the interaction between the subgroups $N$, $U$, and $T$ will be critical in describing the Hecke algebra multiplication in the following section. They can all be found in [Ste67].

The subgroup

$$U = \langle x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q \rangle$$

has generators $\{x_\alpha(t) \mid \alpha \in R^+, t \in \mathbb{F}_q\}$, with relations

$$x_\alpha(a)x_\beta(b)x_\alpha(a)^{-1}x_\beta(b)^{-1} = \prod_{\substack{\gamma = i\alpha + j\beta \in R^+ \\ i,j \in \mathbb{Z}_{>0}}} x_\gamma(z_\gamma a^i b^j), \tag{U1}$$

$$x_\alpha(a)x_\alpha(b) = x_\alpha(a+b), \tag{U2}$$

where $z_\gamma \in \mathbb{Z}$ depends on $i, j, \alpha, \beta$, but not on $a, b \in \mathbb{F}_q$ [Ste67]. The $z_\gamma$ have been explicitly computed for various types in [Dem65, Ste67] (See also Appendix A).

The subgroup $N$ has generators $\{\xi_i,\ h_H(t) \mid i = 1, 2, \ldots, \ell,\ H \in \mathfrak{h}_{\mathbb{Z}},\ t \in \mathbb{F}_q^*\}$, with relations

$$\xi_i^2 = h_i(-1), \tag{N1}$$

$$\underbrace{\xi_i\xi_j\xi_i\xi_j\cdots}_{m_{ij}\text{ terms}} = \underbrace{\xi_j\xi_i\xi_j\xi_i\cdots}_{m_{ij}\text{ terms}}, \qquad \text{where } (s_is_j)^{m_{ij}} = 1 \text{ in } W, \tag{N2}$$

$$\xi_i h_H(t) = h_{s_i(H)}(t)\xi_i, \tag{N3}$$

$$h_H(a)h_H(b) = h_H(ab), \tag{N4}$$

$$h_H(a)h_{H'}(b) = h_{H'}(b)h_H(a), \qquad \text{for } H, H' \in \mathfrak{h}, \tag{N5}$$

$$h_H(a)h_{H'}(a) = h_{H+H'}(a), \qquad \text{for } H, H' \in \mathfrak{h}, \tag{N6}$$

$$h_{H_1}(t_1)h_{H_2}(t_2)\ldots h_{H_k}(t_k) = 1, \qquad \text{if } t_1^{\lambda_j(H_1)}\cdots t_k^{\lambda_j(H_k)} = 1 \text{ for all } 1 \le j \le n, \tag{N7}$$

where $\lambda_j : \mathfrak{h} \to \mathbb{C}$ depends on $V$ as in (2.6).

The double-coset decomposition of $G$ (3.10) implies $G = \langle U, N\rangle$. Thus, $G$ is generated by $\{x_\alpha(a), \xi_i, h_H(b) \mid \alpha \in R^+, a \in \mathbb{F}_q, i = 1, 2, \ldots, \ell, H \in \mathfrak{h}_{\mathbb{Z}}, b \in \mathbb{F}_q^*\}$ with relations (U1)-(N7) and

$$\xi_i x_\alpha(t)\xi_i^{-1} = x_{s_i(\alpha)}(c_{i\alpha}t), \qquad \text{where } c_{i\alpha} = \pm 1\ (c_{i\alpha_i} = -1) \tag{UN1}$$

$$hx_\alpha(b)h^{-1} = x_\alpha(\alpha(h)b), \qquad \text{for } h \in T, \tag{UN2}$$

$$\xi_i x_i(t)\xi_i = x_i(t^{-1})h_i(t^{-1})\xi_i x_i(t^{-1}), \qquad \text{where } x_i(t) = x_{\alpha_i}(t) \text{ and } t \neq 0. \tag{UN3}$$

where for $\alpha \in R$ and $h_H(t) \in T$,

$$\alpha(h_H(t)) = t^{\alpha(H)}. \tag{3.15}$$

Fix a $\psi_\mu : U \to \mathbb{C}^*$ as in (3.7). For $k \in \mathbb{F}_q$, let

$$e_\alpha(k) = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt)x_\alpha(t) \quad \text{with the convention} \quad e_\alpha = e_\alpha(1). \tag{3.16}$$

Note that for any given ordering of the positive roots, the decomposition

$$U = \prod_{\alpha\in R^+} U_\alpha \qquad \text{implies} \qquad e_\mu = \prod_{\alpha\in R^+} e_\alpha. \tag{3.17}$$

In particular, given any $\alpha \in R^+$, we may choose the ordering of the positive roots to have $e_\alpha$ appear either first or last. Therefore, since $e_\alpha$ is an idempotent,

$$e_\mu e_\alpha = e_\mu = e_\alpha e_\mu. \tag{3.18}$$

If $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ with $r$ minimal, then let

$$\begin{aligned} R_w &= \{\alpha \in R^+ \mid w(\alpha) \in R^-\} \\ &= \{\alpha_{i_r}, s_{i_r}(\alpha_{i_{r-1}}), \ldots, s_{i_r}s_{i_{r-1}}\cdots s_{i_2}(\alpha_{i_1})\}, \end{aligned} \tag{3.19}$$

where the second equality is in [Bou02, VI.1, Corollary 2 of Proposition 17].

**Lemma 3.6.** *Let $v \in N$, and let $w = \pi(v)$ (with $\pi : N \to W$ as in (3.3)).*

$$\xi_i e_\alpha(k)\xi_i^{-1} = e_{s_i\alpha}(c_{i\alpha}k), \qquad \text{for } \alpha \in R^+,\ 1 \le i \le n-1, \tag{E1}$$

$$v e_\alpha v^{-1} = e_{w\alpha}, \qquad \text{for } \alpha \notin R_w, v \in N_\mu, \tag{E2}$$

$$h e_\alpha(k) h^{-1} = e_\alpha(k\alpha(h)^{-1}), \qquad \text{for } h \in T, \tag{E3}$$

$$e_\mu x_\alpha(t) = \psi(\mu_\alpha t)e_\mu = x_\alpha(t)e_\mu. \tag{E4}$$

*Proof.* (E1) Using relation (UN1), we have

$$\begin{aligned}\xi_i e_\alpha(k)\xi_i^{-1} &= \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt)\xi_i x_\alpha(t)\xi_i^{-1} = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha kt)x_{s_i\alpha}(c_{i\alpha}t)\\ &= \frac{1}{q}\sum_{t'\in\mathbb{F}_q}\psi(-\mu_\alpha c_{i\alpha}kt')x_{s_i\alpha}(t') = e_{s_i\alpha}(c_{i\alpha}k).\end{aligned}$$

(E2) Suppose $\alpha \notin R_w$. Since $v \in N_\mu$,

$$\begin{aligned}\psi(\mu_\alpha t) = \psi_\mu(x_\alpha(t)) = \psi_\mu(v x_\alpha(t) v^{-1}) &= \psi_\mu(x_{w\alpha}(kt)) && \text{(by (UN1))}\\ &= \psi(\mu_{w\alpha}kt), \quad \text{for some } k =\in \mathbb{Z}_{\neq 0}.\end{aligned}$$

In particular, $\mu_\alpha = k\mu_{w\alpha}$ and we may conclude

$$v e_\alpha v^{-1} = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-\mu_\alpha t)x_{w\alpha}(kt) = \frac{1}{q}\sum_{t'\in\mathbb{F}_q}\psi(-\mu_\alpha k^{-1}t')x_{w\alpha}(t') = e_{w\alpha}.$$

(E3) Since $h x_\alpha(t) h^{-1} = x_\alpha(\alpha(h)t)$, we have

$$h e_\alpha(k) h^{-1} = \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-kt)x_\alpha(\alpha(h)t) = \sum_{t\in\mathbb{F}_q}\psi(-kt\alpha(h)^{-1})x_\alpha(t) = e_\alpha(k\alpha(h)^{-1}).$$

(E4) Note that

$$\begin{aligned}e_\alpha x_\alpha(t) &= \frac{1}{q}\sum_{a\in\mathbb{F}_q}\psi(-\mu_\alpha a)x_\alpha(a)x_\alpha(t) = \frac{1}{q}\sum_{a\in\mathbb{F}_q}\psi(-\mu_\alpha a)x_\alpha(a+t)\\ &= \frac{1}{q}\sum_{a'\in\mathbb{F}_q}\psi(-\mu_\alpha(a'-t))x_\alpha(a') = \frac{1}{q}\sum_{a'\in\mathbb{F}_q}\psi(-\mu_\alpha a')\psi(\mu_\alpha t)x_\alpha(a')\\ &= \psi(\mu_\alpha t)e_\alpha\end{aligned}$$

Therefore, by (3.18), $e_\mu x_\alpha(t) = e_\mu e_\alpha x_\alpha(t) = \psi(\mu_\alpha t)e_\mu$. □

### 3.3.2 Local Hecke algebra relations

Let $u = u_1u_2\cdots u_ru_T \in N$ decompose according to $s_{i_1}s_{i_2}\ldots s_{i_r} \in W$. For $1 \le k \le r$ define constants $c_k = \pm 1$ and roots $\beta_k \in R^+$ by the equation

$$x_{\beta_k}(c_kt) = (u_{k+1}\cdots u_r)^{-1}x_{\alpha_{i_k}}(t)(u_{k+1}\cdots u_r). \tag{3.20}$$

Note that $R_{\pi(u)} = \{\beta_1, \beta_2, \ldots, \beta_r\}$ (see (3.19)). Define $f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r]$ by

$$f_u = -\frac{\mu_{\beta_1}c_1}{\beta_1(u_T)}y_1 - \frac{\mu_{\beta_2}c_2}{\beta_2(u_T)}y_2 - \cdots - \frac{\mu_{\beta_r}c_r}{\beta_r(u_T)}y_r. \tag{3.21}$$

and for $k = 1, 2, \ldots, r$, let

$$u_k(t) = \xi_{i_k}(t), \qquad \text{where} \quad \xi_i(t) = \xi_i x_i(t). \tag{3.22}$$

**Theorem 3.7.** *Let $u = u_1u_2\cdots u_ru_T, v = v_1v_2\cdots v_sv_T \in N_\mu$ decompose according to $s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ and $s_{j_1}s_{j_2}\cdots s_{j_s} \in W$, respectively, as in (3.4). Then*

*(a)*

$$(e_\mu ue_\mu)(e_\mu ve_\mu) = \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f_u)(t)\, e_\mu(u_1(t_1)u_2(t_2)\cdots u_r(t_r))(v_1v_2\cdots v_s)he_\mu,$$

*where $h = v_Tv^{-1}u_Tv \in T$.*

*(b) The following local relations suffice to compute the product $(e_\mu ue_\mu)(e_\mu ve_\mu)$.*

$$\sum_{t\in\mathbb{F}_q}(\psi\circ f)(t)\xi_i(t)\xi_i = (\psi\circ f)(0)\xi_i^2 + \sum_{t\in\mathbb{F}_q^*}(\psi\circ f)(t)x_i(t^{-1})\xi_ih_i(t)x_i(t^{-1}), \tag{$\mathcal{H}1$}$$

$$\xi_ix_\alpha(t) = x_{s_i(\alpha)}(c_{i\alpha}t)\xi_i, \tag{$\mathcal{H}2$}$$

$$x_\alpha(t)h = hx_\alpha(\alpha(h)^{-1}t), \tag{$\mathcal{H}3$}$$

$$e_\mu x_\alpha(t) = \psi(\mu_\alpha t)e_\mu = x_\alpha(t)e_\mu, \tag{$\mathcal{H}4$}$$

$$(\psi\circ f)(t)(\psi\circ g)(t) = (\psi\circ(f+g))(t), \quad \text{for } f, g \in \mathbb{F}_q[y_1^{\pm1}, \ldots, y_r^{\pm1}],\ t \in \mathbb{F}_q^r, \tag{$\mathcal{H}5$}$$

$$h_\alpha(t)\xi_i = \xi_ih_{s_i(\alpha)}(t), \tag{$\mathcal{H}6$}$$

$$\xi_i(a)x_\alpha(b) = \prod_{\substack{\gamma=m\alpha_i+n\alpha\in R^+\\ m,n\ge0}}x_{s_i\gamma}(c_{is_i(\gamma)}z_\gamma a^mb^n)\xi_i(a), \quad \text{where } \alpha \ne \alpha_i, \tag{$\mathcal{H}7$}$$

$$\sum_{a\in\mathbb{F}_q}\Phi(a)\xi_i(a)x_i(b) = \sum_{a\in\mathbb{F}_q}\Phi(a-b)\xi_i(a), \quad \text{for some map } \Phi : \mathbb{F}_q \to \mathbb{C}G, \tag{$\mathcal{H}8$}$$

$$h_\alpha(a)h_\alpha(b) = h_\alpha(ab), \tag{$\mathcal{H}9$}$$

$$\underbrace{\xi_i\xi_j\xi_i\xi_j\cdots}_{m_{ij}\text{ terms}} = \underbrace{\xi_j\xi_i\xi_j\xi_i\cdots}_{m_{ij}\text{ terms}}, \quad \text{where } m_{ij} \text{ is the order of } s_is_j \text{ in } W. \tag{$\mathcal{H}10$}$$

*Proof.* **(a)** Order the positive roots so that by (3.17)

$$
\begin{aligned}
e_\mu u e_\mu v e_\mu &= e_\mu u\Big(\prod_{\alpha\notin R_w} e_\alpha\Big) e_{\beta_1} e_{\beta_2}\cdots e_{\beta_r} v e_\mu && \text{(definition } \beta_k)\\
&= e_\mu\Big(\prod_{\alpha\notin R_w} e_{w\alpha}\Big) u e_{\beta_1} e_{\beta_2}\cdots e_{\beta_r} v e_\mu && \text{(Lemma 3.6,E2)}\\
&= e_\mu u e_{\beta_1} e_{\beta_2}\cdots e_{\beta_r} v e_\mu && \text{(Lemma 3.6, E4)}\\
&= e_\mu u_1 u_2\cdots u_r u_T e_{\beta_1} e_{\beta_2}\cdots e_{\beta_r} v e_\mu\\
&= e_\mu u_1 u_2\cdots u_r e_{\beta_1}(\tfrac{\mu_{\beta_1}}{\beta_1(u_T)}) e_{\beta_2}(\tfrac{\mu_{\beta_2}}{\beta_2(u_T)})\cdots e_{\beta_r}(\tfrac{\mu_{\beta_r}}{\beta_r(u_T)}) u_T v e_\mu && \text{(Lemma 3.6,E3)}\\
&= e_\mu u_1 e_{\alpha_{i_1}}(\tfrac{\mu_{\beta_1} c_1}{\beta_1(u_T)}) u_2 e_{\alpha_{i_2}}(\tfrac{\mu_{\beta_2} c_2}{\beta_2(u_T)})\cdots u_r e_{\alpha_{i_r}}(\tfrac{\mu_{\beta_r} c_r}{\beta_r(u_T)}) u_T v e_\mu && \text{(Lemma 3.6,E1)}\\
&= e_\mu u_1 e_{\alpha_{i_1}}(\tfrac{\mu_{\beta_1} c_1}{\beta_1(u_T)}) u_2 e_{\alpha_{i_2}}(\tfrac{\mu_{\beta_2} c_2}{\beta_2(u_T)})\cdots u_r e_{\alpha_{i_r}}(\tfrac{\mu_{\beta_r} c_r}{\beta_r(u_T)}) v_1\cdots v_s v_T v^{-1} u_T v e_\mu\\
&= \frac{e_\mu}{q^r}\sum_{t_1,\dots,t_r\in\mathbb{F}_q}\psi(-\tfrac{\mu_{\beta_1} c_1 t_1}{\beta_1(u_T)}) u_1(t_1)\cdots\psi(-\tfrac{\mu_{\beta_r} c_r t_r}{\beta_r(u_T)}) u_r(t_r) v_1\cdots v_s h e_\mu && \text{(definition } e_\alpha)\\
&= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f_u)(t) e_\mu u_1(t_1)\cdots u_r(t_r) v_1\cdots v_s h e_\mu, && \text{(by } (\mathcal{H}5))
\end{aligned}
$$

where $h = v_T v^{-1} u_T v \in T$, as desired.

**(b)** First, note that these relations are in fact correct (though not necessarily sufficient): ($\mathcal{H}1$) comes from (UN3); ($\mathcal{H}2$) comes from (UN1); ($\mathcal{H}3$) comes from (UN2); ($\mathcal{H}4$) comes from (E4); ($\mathcal{H}5$) comes from the multiplicativity of $\psi$; ($\mathcal{H}6$) comes from (N3); ($\mathcal{H}7$) comes from (U1) and (UN1); ($\mathcal{H}8$) comes from (U2); ($\mathcal{H}9$) is (N4); and ($\mathcal{H}10$) is (N2). It therefore remains to show sufficiency.

By (a) we may write

$$
(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f)(t) e_\mu u_1(t_1)\cdots u_r(t_r) v_1\cdots v_s h e_\mu
$$

for some $f \in \mathbb{F}_q[y_1,\dots,y_r]$ and $h \in T$. Say $t_k$ is *resolved* if the only part of the sum depending on $t_k$ is $(\psi\circ f)$. The product is *reduced* when all the $t_k$ are resolved. I will show how to resolve $t_r$ and the result will follow by induction.

Use relation ($\mathcal{H}2$) to define the constant $d$ and the root $\gamma \in R$ by

$$
(v_1 v_2\cdots v_s)^{-1} x_{\alpha_{i_r}}(t)(v_1 v_2\cdots v_s) = x_\gamma(dt) \qquad (\text{where } \ell(\pi(v)) = s). \tag{3.23}
$$

There are two possible situations:

**Case 1.** $\gamma \in R^+$,

**Case 2.** $\gamma \in R^-$.

In Case 1,

$$
\begin{aligned}
(e_\mu u e_\mu)(e_\mu v e_\mu) &= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_r\underrightarrow{x_{i_r}(t_r)}v_1\cdots v_s h e_\mu && \text{(by (a))}\\
&= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s\underrightarrow{x_\gamma(dt_r)}h e_\mu && \text{(by } (\mathcal{H}2))\\
&= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h\underline{x_\gamma(d\gamma(h)^{-1}t_r)e_\mu} && \text{(by } (\mathcal{H}3))\\
&= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h\underline{\psi(\mu_\gamma d\gamma(h)^{-1}t_r)}e_\mu && \text{(by } (\mathcal{H}4))\\
&= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h\underleftarrow{(\psi\circ\mu_\gamma d\gamma(h)^{-1}y_r)(t_r)}e_\mu\\
&= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^r}(\psi\circ g)(t)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h e_\mu, && \text{(by } (\mathcal{H}5))
\end{aligned}
$$

where $g = f + \mu_\gamma d\gamma(h)^{-1}y_r$. We have resolved $t_r$ in Case 1.

In Case 2, $\gamma\in R^-$, so we can no longer move $x_{i_r}(t_r)$ past the $v_j$. Instead,

$$
\begin{aligned}
&(e_\mu u e_\mu)(e_\mu v e_\mu)\\
&= \frac{e_\mu}{q^r}\sum_{t\in\mathbb{F}_q^r}\underline{(\psi\circ f)(t)}u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r x_{i_r}(t_r)u_r u_r^{-1}v_1\cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}u_1(t_1)\cdots u_{r-1}(t_{r-1})\sum_{t_r\in\mathbb{F}_q}(\psi\circ f)(t',t_r)\underline{u_r x_{i_r}(t_r)u_r}u_r^{-1}v_1\cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}u_1(t_1)\cdots u_{r-1}(t_{r-1})(\psi\circ f)(t',0)\underline{u_r^2 u_r^{-1}}v_1\cdots v_s h e_\mu && \text{(by } (\mathcal{H}1))\\
&+\frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}u_1(t_1)\cdots u_{r-1}(t_{r-1})\sum_{t_r\in\mathbb{F}_q^*}(\psi\circ f)(t',t_r)x_{i_r}(t_r^{-1})h_{i_r}(t_r^{-1})u_r\underrightarrow{x_{i_r}(t_r^{-1})}u_r^{-1}v_1\cdots v_s h e_\mu\\
&= \frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}u_1(t_1)\cdots u_{r-1}(t_{r-1})\underleftarrow{(\psi\circ f)(t',0)}u_r v_1\cdots v_s h e_\mu\\
&\quad+\frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}u_1(t_1)\cdots u_{r-1}(t_{r-1})\sum_{t_r\in\mathbb{F}_q^*}(\psi\circ g)(t',t_r)x_{i_r}(t_r^{-1})\underrightarrow{h_{i_r}(-t_r)}v_1\cdots v_s h e_\mu,\\
& && \text{(by } (\mathcal{H}2,\mathcal{H}3,\mathcal{H}4))
\end{aligned}
$$

where $g = f - \mu_\gamma d\gamma(h)^{-1}y_r^{-1}$ (same as in the analogous steps in Case 1).

$$\begin{aligned}
&= \frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}(\psi\circ f)(t',0)u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h e_\mu \\
&\quad + \frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}u_1(t_1)\cdots u_{r-1}(t_{r-1})\sum_{t_r\in\mathbb{F}_q^*}\underleftarrow{(\psi\circ g)(t',t_r)x_{i_r}(t_r^{-1})}v_1\cdots v_s h_\gamma(-t_r)he_\mu, \quad \text{(by } (\mathcal{H}6))\\
&= \frac{e_\mu}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}(\psi\circ f)(t',0)u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h e_\mu \\
&\quad + \frac{e_\mu}{q^r}\sum_{\substack{t'\in\mathbb{F}_q^{r-1}\\ t_r\in\mathbb{F}_q^*}}(\psi\circ\varphi(g))(t',t_r)\Big(\prod_{\beta\in R^+}\underleftarrow{x_\beta(a_\beta(t,t_r))}\Big)u_1(t_1)\cdots u_{r-1}(t_{r-1})v_1\cdots v_s h' e_\mu \\
&\hspace{30em}\text{(by } (\mathcal{H}7,\mathcal{H}8))
\end{aligned}$$

where $\varphi : \mathbb{F}_q[y_1,\ldots,y_r] \to \mathbb{F}_q[y_1,\ldots,y_r]$ catalogues the substitutions to $g$ due to $(\mathcal{H}8)$, the $a_\beta(y_1,y_2,\ldots,y_r^{-1}) \in \mathbb{F}_q[y_1,\ldots,y_{r-1},y_r]$ are determined by repeated applications of $(\mathcal{H}7)$ and $(\mathcal{H}8)$, and $h' = h_\gamma(-t_r)h \in T$.

$$\begin{aligned}
&= \frac{1}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}(\psi\circ f)(t',0)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h e_\mu \\
&\quad + \frac{1}{q^r}\sum_{\substack{t'\in\mathbb{F}_q^{r-1}\\ t_r\in\mathbb{F}_q^*}}(\psi\circ\varphi(g))(t',t_r)e_\mu\Big(\prod_{\beta\in R^+}\underleftarrow{\psi(\mu_\beta a_\beta(t,t_r))}\Big)u_1(t_1)\cdots u_{r-1}(t_{r-1})v_1\cdots v_s h' e_\mu \\
&\hspace{30em}\text{(by } (\mathcal{H}4))\\
&= \frac{1}{q^r}\sum_{t'\in\mathbb{F}_q^{r-1}}(\psi\circ f)(t',0)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})u_r v_1\cdots v_s h e_\mu \\
&\quad + \frac{1}{q^r}\sum_{\substack{t'\in\mathbb{F}_q^{r-1}\\ t_r\in\mathbb{F}_q^*}}(\psi\circ g_2)(t',t_r)e_\mu u_1(t_1)\cdots u_{r-1}(t_{r-1})v_1\cdots v_s h' e_\mu, \quad \text{(by } (\mathcal{H}5))
\end{aligned}$$

where $g_2 = \varphi(g) + \sum_{\beta\in R^+}\mu_\beta a_\beta(y_1,\ldots,y_{r-1},y_r^{-1})$. Now $t_r$ is resolved for Case 2, as desired. □

**Lemma 3.8 (Resolving $t_k$).** *Let $u = u_1u_2\cdots u_k \in N$ decompose according to $s_{i_1}s_{i_2}\cdots s_{i_k}$ (with $u_T = 1$). Suppose $v \in N$ and $f \in \mathbb{F}_q[y_1,y_2,\ldots,y_k]$. define $\gamma \in R$ and $d \in \mathbb{C}$ by the equation $v^{-1}x_{i_k}(t)v = x_\gamma(dt)$. Then*

**Case 1** *If $\ell(\pi(u_k v)) > \ell(\pi(v))$, then*

$$\sum_{t\in\mathbb{F}_q^k}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_k(t_k)ve_\mu = \sum_{t\in\mathbb{F}_q^k}(\psi\circ(\underline{f+\mu_\gamma dc_\gamma y_k}))(t)e_\mu u_1(t_1)\cdots u_{k-1}(t_{k-1})\underline{u_k}ve_\mu.$$

**Case 2** *If $\ell(\pi(u_k v)) < \ell(\pi(v))$, then*

$$\sum_{t\in\mathbb{F}_q^k}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_k(t_k)ve_\mu = \sum_{\substack{t\in\mathbb{F}_q^k,\\ \underline{t_k=0}}}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_{k-1}(t_{k-1})\underline{u_k}ve_\mu + \sum_{\substack{t\in\mathbb{F}_q^k,\\ \underline{t_k\in\mathbb{F}_q^*}}}(\psi\circ(\underline{\varphi_k(f)-\mu_\gamma dy_k^{-1}}))(t)e_\mu u_1(t_1)\cdots u_{k-1}(t_{k-1})\underline{h_{i_k}(t_k^{-1})}ve_\mu,$$

*where $\varphi_k : \mathbb{F}_q[y_1^{\pm 1},\dots,y_k^{\pm 1}] \to \mathbb{F}_q[y_1^{\pm 1},\dots,y_k^{\pm 1}]$ is given by*

$$\sum_{\substack{t\in\mathbb{F}_q^k\\ t_k\in\mathbb{F}_q^*}}(\psi\circ f)(t)e_\mu u_1(t_1)\cdots u_{k-1}(t_{k-1})\underleftarrow{x_{i_k}(t_k^{-1})} = \sum_{\substack{t\in\mathbb{F}_q^k\\ t_k\in\mathbb{F}_q^*}}(\psi\circ\underline{\varphi_k(f)})(t)e_\mu u_1(t_1)\cdots u_{k-1}(t_{k-1}).$$

*Proof.* This Lemma puts $v$ in the place of $u_T v$ in the proof of Theorem 3.7, (b), and summarizes the steps taken in Case 1 and Case 2. □

### 3.3.3 Global Hecke algebra relations

Fix $u = u_1u_2\cdots u_r u_T \in N_\mu$, decomposed according $s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ (see (3.4)). Suppose $v' \in N_\mu$ and let $v = u_T v'$.

For $0 \le k \le r$, let $\tau = (\tau_1, \tau_2, \dots, \tau_{r-k})$ be such that $\tau_i \in \{+0, -0, 1\}$, where $+0$, $-0$, and 1 are symbols. If $\tau$ has $r-k$ elements, then the *colength* of $\tau$ is $\ell^\vee(\tau) = k$. For example, if $r = 10$ and $\tau = (-0, 1, +0, +0, 1, 1)$, then $\ell^\vee(\tau) = 4$. For $i \in \{+0, -0, 1\}$, let

$$(i,\tau) = (i, \tau_1, \tau_2, \cdots, \tau_{r-k}).$$

By convention, if $\ell^\vee(\tau) = r$, then $\tau = \emptyset$.

Suppose $\ell^\vee(\tau) = k$. Define

$$\Xi^\tau(u,v) = \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^\tau}(\psi\circ f^\tau)(t)e_\mu u_1(t_1)\cdots u_k(t_k)v^\tau(t)e_\mu, \tag{3.24}$$

where

$$\mathbb{F}_q^\tau = \left\{ t \in \mathbb{F}_q^r \;\middle|\; \text{for } k < i \le r, \begin{array}{l} \text{if } \tau_{i-k} = +0, \text{ then } t_i \in \mathbb{F}_q, \\ \text{if } \tau_{i-k} = -0, \text{ then } t_i = 0, \\ \text{if } \tau_{i-k} = 1, \text{ then } t_i \in \mathbb{F}_q^*. \end{array} \right\}; \tag{3.25}$$

$$v^\tau(t) = h_{i_{k+1}}(t_{k+1})^{\tau_1} u_{k+1}^{1-\tau_1} \cdots h_{i_r}(t_r)^{\tau_{r-k}} u_r^{1-\tau_{r-k}} v, \tag{3.26}$$

with $+0 = -0 = 0 \in \mathbb{Z}$, $1 = 1 \in \mathbb{Z}$ in (3.26); and $f^\tau$ is defined recursively by

$$f^\emptyset = f_u = -\frac{\mu_{\beta_1} c_1}{\beta_1(u_T)} y_1 - \frac{\mu_{\beta_2} c_2}{\beta_2(u_T)} y_2 - \cdots - \frac{\mu_{\beta_r} c_r}{\beta_r(u_T)} y_r, \quad \text{(as in (3.21))}, \tag{3.27}$$

$$f^{(i,\tau)} = \begin{cases} f^\tau + \mu_{\gamma_\tau} d_\tau y_k, & \text{if } i = \pm 0, \\ \varphi_k(f^\tau) - \mu_{\gamma_\tau} d_\tau y_k^{-1}, & \text{if } i = 1, \end{cases} \tag{3.28}$$

where $(v^\tau)^{-1} x_{\alpha_{i_k}}(t) v^\tau = x_{\gamma_\tau}(d_\tau t)$ and the map $\varphi_k$ is as in Lemma 3.8, Case 2.

**Remarks**.

1. By (3.24) and Theorem 3.7 (a), $\Xi^\emptyset(u,v) = (e_\mu u e_\mu)(e_\mu v' e_\mu)$ (recall, $v = u_T v'$).
2. If $\ell^\vee(\tau) = 0$ so that $\tau$ is a string of length $r$, then
   (a) $\Xi^\tau(u,v) = \dfrac{1}{q^r} \displaystyle\sum_{t \in \mathbb{F}_q^\tau} (\psi \circ f^\tau)(t) e_\mu v^\tau e_\mu$ has no remaining factors of the form $u_k(t_k)$,
   (b) $\Xi^\tau(u,v) = 0$ unless $v^\tau(t) \in N_\mu$ for some $t \in \mathbb{F}_q$.

The following corollary gives relations for expanding $\Xi^\tau(u,v)$ (beginning with $\Xi^\emptyset(u,v)$) as a sum of terms of the form $\Xi^{\tau'}$ with $\ell^\vee(\tau') = \ell^\vee(\tau) - 1$. When each term has colength 0 (or length $r$), then we will have decomposed the product $(e_\mu u e_\mu)(e_\mu v' e_\mu)$ in terms of the basis elements of $\mathcal{H}_\mu$.

In summary, while we compute $f^\tau$ recursively by *removing* elements from $\tau$, we compute the product $(e_\mu u e_\mu)(e_\mu v' e_\mu)$ by progressively *adding* elements to $\tau$.

**Corollary 3.9 (The Global Alternative).** *Let $u, v' \in N_\mu$ such that $u = u_1 u_2 \cdots u_r u_T$ decomposes according to a minimal expression in $W$. Let $v = u_T v'$. Then*

*(a)* $(e_\mu u e_\mu)(e_\mu v' e_\mu) = \Xi^\emptyset(u,v)$

*(b) If $\ell^\vee(\tau) = k$, then*

$$\Xi^\tau(u,v) = \begin{cases} \Xi^{(+0,\tau)}(u,v), & \text{if } \ell(\pi(u_k v^\tau)) > \ell(\pi(v^\tau)), \\ \Xi^{(-0,\tau)}(u,v) + \Xi^{(1,\tau)}(u,v), & \text{if } \ell(\pi(u_k v^\tau)) < \ell(\pi(v^\tau)). \end{cases}$$

*Proof.* (a) follows from Remark 1.
(b) Suppose $\ell^\vee(\tau) = k$. Note that

$$\begin{aligned}\Xi^\tau(u,v) &= \frac{1}{q^r}\sum_{t\in\mathbb{F}_q^\tau}(\psi\circ f^\tau)(t)e_\mu u_1(t_1)\cdots u_k(t_k)v^\tau e_\mu \\ &= \frac{1}{q^r}\sum_{t''\in(\mathbb{F}_q^{r-k})^\tau}\sum_{t'\in\mathbb{F}_q^k}(\psi\circ f^\tau)(t',t'')e_\mu u_1(t_1)\cdots u_k(t_k)v^\tau e_\mu\end{aligned}$$

where $(\mathbb{F}_q^{r-k})^\tau = \{(t_{k+1},\ldots,t_r)\in\mathbb{F}_q^{r-k} \mid \text{restrictions according to } \tau\}$ (as in (3.25)). Apply Lemma 3.8 to the inside sum with $f := f^\tau$, $v := v^\tau$. Note that the Lemma relations imply

$$\{t'\in\mathbb{F}_q^k\} \quad \text{becomes} \quad \begin{cases}\{t'\in\mathbb{F}_q^k\}, & \text{if in Case 1,}\\ \{t'\in\mathbb{F}_q^k \mid t_k = 0\}, & \text{if in Case 2, first sum,}\\ \{t'\in\mathbb{F}_q^k \mid t_k\in\mathbb{F}_q^*\}, & \text{if in Case 2, second sum,}\end{cases}$$

$$f^\tau \quad \text{becomes} \quad \begin{cases} f^{(+0,\tau)}, & \text{if in Case 1,}\\ f^{(-0,\tau)}, & \text{if in Case 2, first sum,}\\ f^{(1,\tau)}, & \text{if in Case 2, second sum.}\end{cases}$$

$$v^\tau \quad \text{becomes} \quad \begin{cases} v^{(+0,\tau)}, & \text{if in Case 1,}\\ v^{(-0,\tau)}, & \text{if in Case 2, first sum,}\\ v^{(1,\tau)}, & \text{if in Case 2, second sum.}\end{cases}$$

Thus,

$$\Xi^\tau(u,v) = \begin{cases}\Xi^{(+0,\tau)}(u,v), & \text{if Case 1,}\\ \Xi^{(-0,\tau)}(u,v) + \Xi^{(1,\tau)}(u,v), & \text{if Case 2,}\end{cases}$$

as desired. □

# Chapter 4

# A basis with multiplication in the $G = GL_n(\mathbb{F}_q)$ case

## 4.1 Unipotent Hecke algebras

### 4.1.1 The group $GL_n(\mathbb{F}_q)$

Let $G = GL_n(\mathbb{F}_q)$ be the general linear group over the finite field $\mathbb{F}_q$ with $q$ elements. Define the subgroups

$$\begin{gathered} T = \left\{ \begin{matrix} \text{diagonal} \\ \text{matrices} \end{matrix} \right\}, \quad N = \left\{ \begin{matrix} \text{monomial} \\ \text{matrices} \end{matrix} \right\}, \\ W = \left\{ \begin{matrix} \text{permutation} \\ \text{matrices} \end{matrix} \right\}, \quad \text{and} \quad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}, \end{gathered} \tag{4.1}$$

where a monomial matrix is a matrix with exactly one nonzero entry in each row and column ($N = WT$). If necessary, specify the size of the matrices by a subscript such as $G_n$, $U_n$, $W_n$, etc. If $a \in G_m$ and $b \in G_n$ are matrices, then let $a \oplus b \in G_{m+n}$ be the block diagonal matrix

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Let $x_{ij}(t) \in U$ be the matrix with $t$ in position $(i, j)$, ones on the diagonal and zeroes elsewhere; write $x_i(t) = x_{i,i+1}(t)$. Let $h_{\varepsilon_i}(t) \in T$ denote the diagonal matrix with $t$ in the $i$th slot and ones elsewhere, and let $s_i \in W \subseteq N$ be the identity matrix with the $i$th and $(i+1)$st columns interchanged. That is,

$$\begin{gathered} x_i(t) = Id_{i-1} \oplus \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \oplus Id_{n-i-1}, \quad h_{\varepsilon_i}(t) = Id_{i-1} \oplus (t) \oplus Id_{n-i}, \\ s_i = Id_{i-1} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus Id_{n-i-1}, \end{gathered} \tag{4.2}$$

where $Id_k$ is the $k \times k$ identity matrix. Then

$$\begin{gathered} W = \langle s_1, s_2, \ldots s_{n-1} \rangle, \qquad T = \langle h_{\varepsilon_i}(t) \mid 1 \le i \le n, t \in \mathbb{F}_q^* \rangle, \qquad N = WT, \\ U = \langle x_{ij}(t) \mid 1 \le i < j \le n, t \in \mathbb{F}_q \rangle, \qquad G = \langle U, W, T \rangle. \end{gathered} \tag{4.3}$$

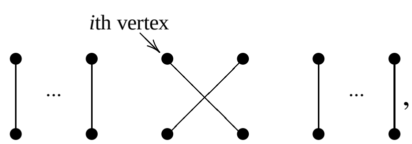
The Chevalley group relations for $G$ are (see also Section 3.3.1)

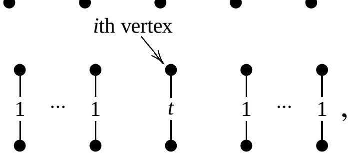
$$x_{ij}(a)x_{rs}(b) = x_{ij}(b)x_{rs}(a)x_{is}(\delta_{jr}ab)x_{rj}(-\delta_{is}ab), \qquad (i,j) \neq (r,s), \tag{U1}$$
$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b), \tag{U2}$$
$$s_i^2 = 1, \tag{N1}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{and} \quad s_i s_j = s_j s_i, \qquad |i-j| > 1, \tag{N2}$$
$$h_{\varepsilon_i}(b)h_{\varepsilon_i}(a) = h_{\varepsilon_i}(ab), \tag{N4}$$
$$h_{\varepsilon_j}(b)h_{\varepsilon_i}(a) = h_{\varepsilon_j}(b)h_{\varepsilon_i}(a), \tag{N5}$$
$$s_r x_{ij}(t) = x_{s_r(i)s_r(j)}(t)s_r, \tag{UN1}$$
$$x_{ij}(a)h_{\varepsilon_r}(t) = h_{\varepsilon_r}(t)x_{ij}(t^{-\delta_{ri}}t^{\delta_{rj}}a), \tag{UN2}$$
$$s_i x_i(t) s_i = x_i(t^{-1})s_i x_i(-t)h_{\varepsilon_i}(t)h_{\varepsilon_{i+1}}(-t^{-1}), \qquad t \neq 0, \tag{UN3}$$

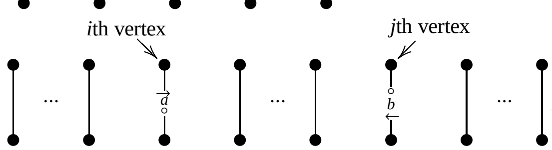
where $\delta_{ij}$ is the Kronecker delta.

### 4.1.2 A pictorial version of $GL_n(\mathbb{F}_q)$

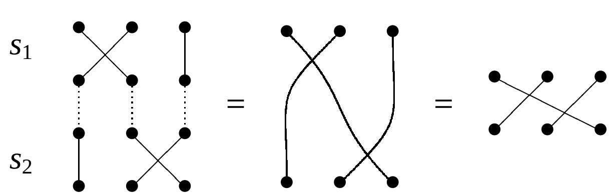
For the results that follow, it will be useful to view these elements of $\mathbb{C}G$ as braid-like diagrams instead of matrices. Consider the following depictions of elements by diagrams with vertices, strands between the vertices, and various objects that slide around on the strands. View

$s_i$ as  (4.4)

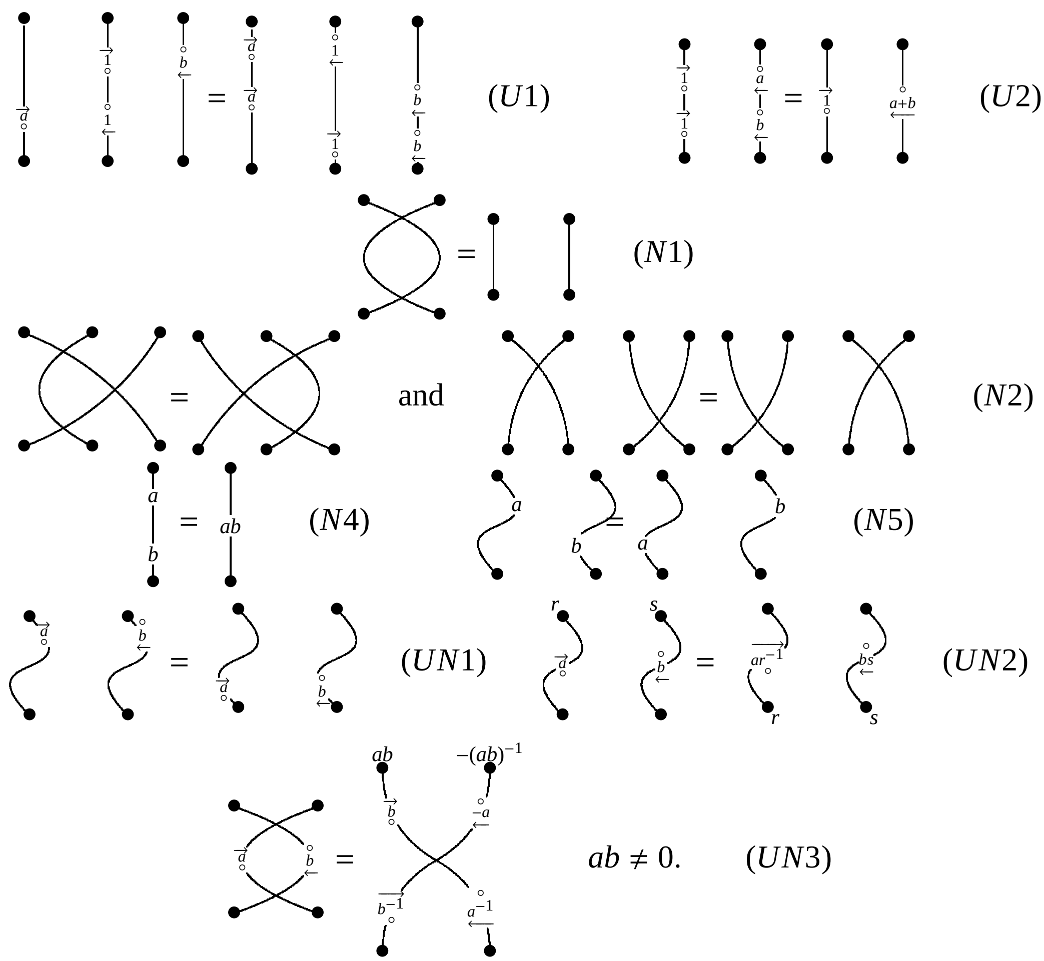

$h_{\varepsilon_i}(t)$ as  (4.5)


$x_{ij}(ab)$ as  (4.6)


where each diagram has two rows of $n$ vertices. Multiplication in $G$ corresponds to the concatenation of two diagrams; for example, $s_2s_1$ is

$s_2s_1 =$ 

In the following Chevalley relations, curved strands indicate longer strands, so for example (UN1) indicates that $\overrightarrow{a}$ and $\overleftarrow{b}$ slide along the strands they are on (no matter how long). The Chevalley relations are




### 4.1.3 The unipotent Hecke algebra $\mathcal{H}_\mu$

Fix a nontrivial group homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$ , and a map

$$\begin{array}{rcl} \mu : \{(i,j) \mid 1 \le i < j \le n\} & \longrightarrow & \mathbb{F}_q \\ (i,j) & \mapsto & \mu_{ij} \end{array} \quad \text{with } \mu_{ij} = 0 \text{ for } j \neq i+1. \tag{4.7}$$

Then

$$\begin{array}{rccc} \psi_\mu : & U & \longrightarrow & \mathbb{C}^* \\ & x_{ij}(t) & \mapsto & \psi(\mu_{ij} t) \end{array} \tag{4.8}$$

is a group homomorphism. Since $\mu_{ij} = 0$ for all $j \neq i+1$, write

$$\mu = (\mu_{(1)}, \mu_{(2)}, \ldots, \mu_{(n-1)}, \mu_{(n)}), \quad \text{where } \mu_{(i)} = \mu_{i,i+1} \text{ and } \mu_{(n)} = 0. \tag{4.7}$$

The unipotent Hecke algebra $\mathcal{H}_\mu$ of the triple $(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = \mathrm{End}_G\left(\mathrm{Ind}_U^G(\psi_\mu)\right) \cong e_\mu \mathbb{C} G e_\mu, \quad \text{where} \quad e_\mu = \frac{1}{|U|}\sum_{u\in U} \psi_\mu(u^{-1})u. \tag{4.9}$$

The *type* of a linear character $\psi_\mu : U \to \mathbb{C}^*$ is the composition $\nu$ such that

$$\mu_{(i)} = 0 \quad \text{if and only if} \quad i \in \nu_\leq \quad \text{(with } \nu_\leq \text{ as in Section 2.3.2).}$$

**Proposition 4.1.** *Let $\nu$ be a composition. Suppose $\chi$ and $\chi'$ are two type $\nu$ linear characters of $U$. Then*

$$\mathcal{H}(G, U, \chi) \cong \mathcal{H}(G, U, \chi').$$

*Proof.* Note that for $h = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in T$,

$$h x_{ij}(t) h^{-1} = x_{ij}(h_i h_j^{-1} t) \qquad \text{for } 1 \leq i < j \leq n, t \in \mathbb{F}_q.$$

Thus, $T$ normalizes $U_{ij} = \langle x_{ij}(t) \mid t \in \mathbb{F}_q\rangle$ and acts on linear characters of $U$ by

$${}^h\chi(u) = \chi(huh^{-1}), \qquad \text{for } h \in T, u \in U.$$

This action preserves the type of $\chi$. The map

$$\begin{array}{ccc} \mathrm{Ind}_U^G(\chi) \cong \mathbb{C}Ge_\chi & \xrightarrow{\sim} & \mathbb{C}Ge_{\chi'} \cong \mathrm{Ind}_U^G(\chi') \\ ge_\chi & \mapsto & ghe_{\chi'} \end{array} \qquad \text{where } \chi' = {}^h\chi \text{ for } h \in T,$$

is a $G$-module isomorphism, so $\mathcal{H}(G, U, \chi) \cong \mathcal{H}(G, U, {}^h\chi)$. It therefore suffices to prove that $T$ acts transitively on the linear characters of type $\mu$.

By Proposition 3.2 (c), the number of $T$-orbits of type $\mu$ linear characters is

$$\begin{aligned}\frac{|Z_\mu|(q-1)^{n-\ell(\mu)}}{|T|} &= \frac{|\{h \in T \mid hx_i(t)h^{-1} = x_i(t), \mu_{(i)} \neq 0\}|(q-1)^{n-\ell(\mu)}}{(q-1)^n} \\ &= \frac{(q-1)^{\ell(\mu)}(q-1)^{n-\ell(\mu)}}{(q-1)^n} \\ &= 1.\end{aligned}$$

Therefore $T$ acts transitively on the type $\mu$ linear characters of $U$. □

Proposition 4.1 implies that given any fixed character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$, it suffices to specify the type of the linear character $\psi_\mu$. Note that the map (given by example)

$$\left\{ \begin{array}{c} \text{Compositions} \\ \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \models n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mu = (\mu_{(1)}, \mu_{(2)}, \ldots, \mu_{(n)}) \\ \mu_{(i)} \in \{0,1\} \text{ and } \mu_{(n)} = 0 \end{array} \right\} \tag{4.10}$$

$$\mu = \begin{array}{|c|c|c|c|c|} \hline 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \\ \hline \end{array} \qquad \leftrightarrow \qquad (1,0,1,1,1,1,0,1,1,0,1,1,1,0)$$

is a bijection. In the following sections identify the compositions $\mu = (\mu_1, \ldots, \mu_\ell)$ with the map $\mu = (\mu_{(1)}, \ldots, \mu_{(n)})$ using this bijection.

The classical examples of unipotent Hecke algebras are the Yokonuma algebra $\mathcal{H}_{(1^n)}$ [Yok69b] and the Gelfand-Graev Hecke algebra $\mathcal{H}_{(n)}$ [Ste67].

## 4.2 An indexing for the standard basis of $\mathcal{H}_\mu$

The group $G$ has a double-coset decomposition $G = \bigsqcup_{v \in N} UvU$, so if

$$\begin{aligned} N_\mu &= \{v \in N \mid e_\mu v e_\mu \neq 0\} \\ &= \{v \in N \mid u, vuv^{-1} \in U \text{ implies } \psi_\mu(u) = \psi_\mu(vuv^{-1})\} \end{aligned} \tag{4.11}$$

then the isomorphism in (4.9) implies that the set $\{e_\mu v e_\mu \mid v \in N_\mu\}$ is a basis for $\mathcal{H}_\mu$ [CR81, Prop. 11.30].

Suppose $a \in M_\ell(\mathbb{F}_q[X])$ is an $\ell \times \ell$ matrix with polynomial entries. Let $d(a_{ij})$ be the degree of the polynomial $a_{ij}$. Define the *degree row sums* and the *degree column sums* of $a$ to be the compositions

$$d^{\rightarrow}(a) = (d^{\rightarrow}(a)_1, d^{\rightarrow}(a)_2, \ldots, d^{\rightarrow}(a)_\ell) \quad \text{and} \quad d^{\downarrow}(a) = (d^{\downarrow}(a)_1, d^{\downarrow}(a)_2, \ldots, d^{\downarrow}(a)_\ell),$$

where

$$d^{\rightarrow}(a)_i = \sum_{j=1}^{\ell} d(a_{ij}) \quad \text{and} \quad d^{\downarrow}(a)_j = \sum_{i=1}^{\ell} d(a_{ij}).$$

Let

$$M_\mu = \{a \in M_{\ell(\mu)}(\mathbb{F}_q[X]) \mid d^{\rightarrow}(a) = d^{\downarrow}(a) = \mu, a_{ij} \text{ monic}, a_{ij}(0) \neq 0\}. \tag{4.12}$$

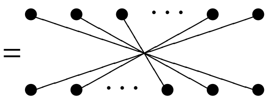
For example,

$$\begin{array}{c} \begin{pmatrix} X+1 & 1 & 1 & X+2 \\ X+3 & X^3+2X+3 & 1 & X+2 \\ 1 & X^2+4X+2 & X+2 & 1 \\ 1 & 1 & X^2+3X+1 & X^2+2 \end{pmatrix} \begin{array}{l} \scriptstyle (1+0+0+1=2) \\ \scriptstyle (1+3+0+1=5) \\ \scriptstyle (0+2+1+0=3) \\ \scriptstyle (0+0+2+2=4) \end{array} \in M_{(2,5,3,4)}. \\ \scriptstyle (1+1+0+0=2) \quad (0+3+2+0=5) \quad (0+0+1+2=3) \quad (1+1+0+2=4) \end{array} \tag{$*$}$$

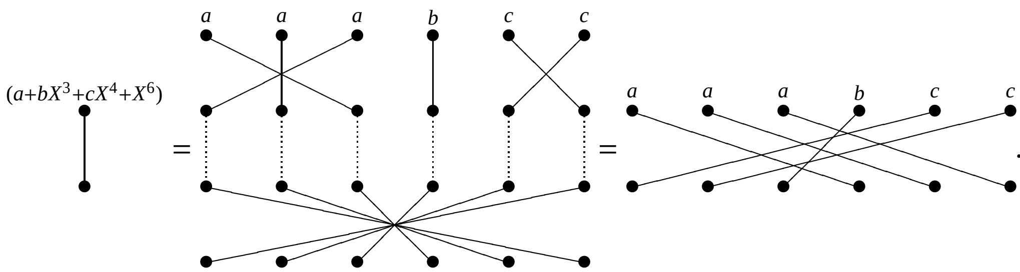
Suppose $(f) = (a_0 + a_1 X^{i_1} + a_2 X^{i_2} + \cdots + a_r X^{i_r} + X^n) \in M_{(n)}$ is a $1 \times 1$ matrix with $a_0, a_1, \ldots, a_r \neq 0$. Let

$(f)$

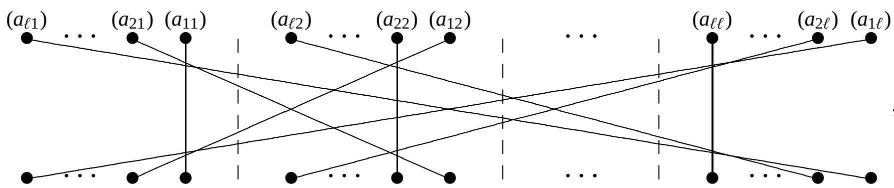
$v_{(f)} =$  $= w_{(n)}(w_{(i_1)} a_0 \oplus w_{(i_2 - i_1)} a_1 \oplus \cdots \oplus w_{(n-i_r)} a_r) \in N,$ (4.13) where

$w_{(k)} =$  $\in W_k,$

and by convention $v_{(1)}$ is the empty strand (not a strand). For example, $v_{(a+bX^3+cX^4+X^6)}$ is

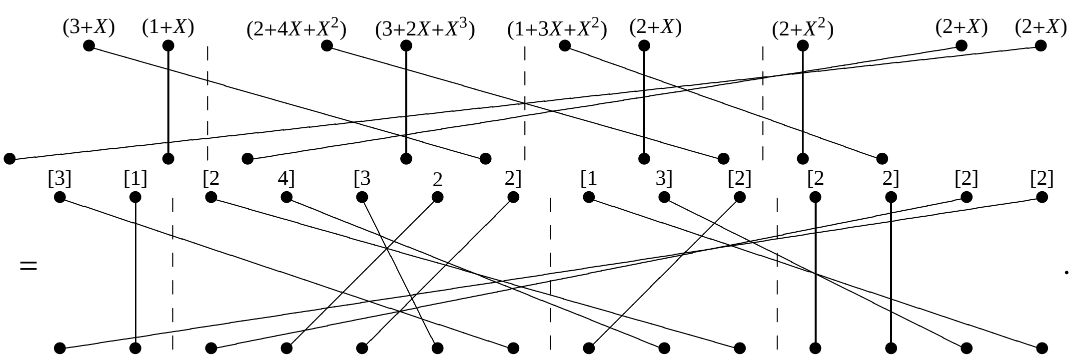



For each $a \in M_\mu$ define a matrix $v_a \in N$ pictorially by

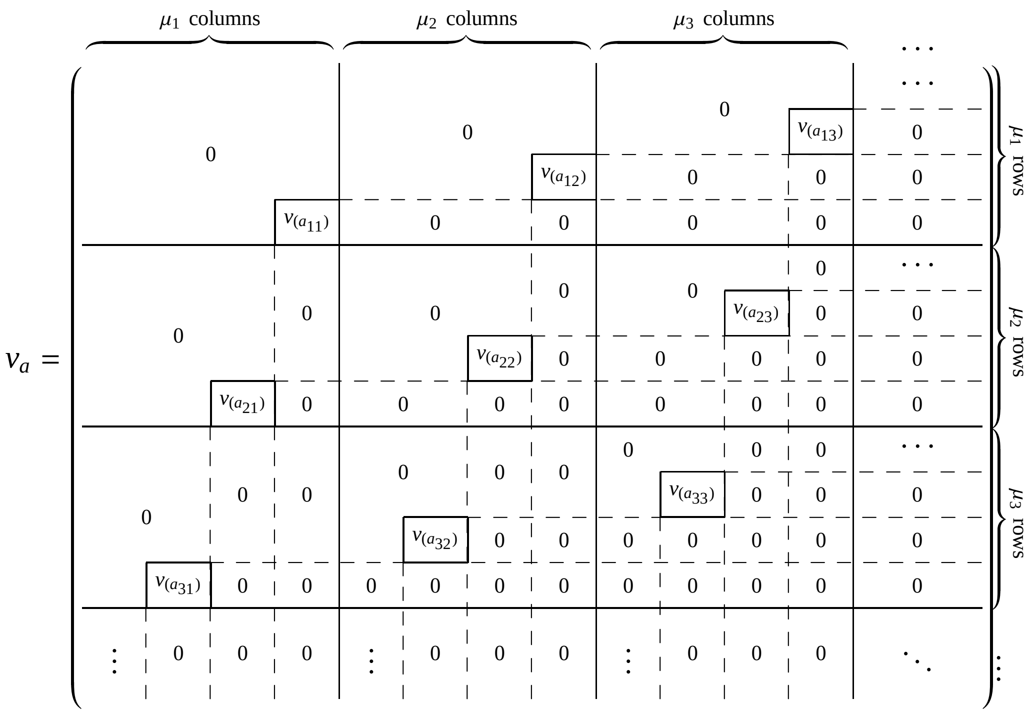
$v_a =$  , (4.14)


where $a_{ij}$ is the $(i, j)$th entry of $a$, and the top vertex associated with $a_{ij}$ goes to the bottom vertex associated with $a_{ji}$. Note that if $a_{ij} = 1$, then both its strand and corresponding vertices vanish. The fact that $a \in M_\mu$ guarantees that we are left with two rows of exactly $n$ vertices.

For example, if $a$ is as in $(*)$, then $v_a$ is



Viewed as a matrix,




**Theorem 4.2.** *Let $N_\mu$ be as in (4.11) and $M_\mu$ be as in (4.12). The map*

$$\begin{array}{ccc} M_\mu & \longrightarrow & N_\mu \\ a & \mapsto & v_a, \end{array}$$

*given by (4.14) is a bijection.*

**Remark.** When $\mu = (n)$ this theorem says that the map $(f) \mapsto v_{(f)}$ of (4.13) is a bijection between $M_{(n)}$ and $N_{(n)}$.

*Proof.* Using the remark following the theorem, it is straightforward to reconstruct $a$ from $v_a$. Therefore the map is invertible, and it suffices to show

(a) the map is well-defined ($v_a \in N_\mu$),

(b) the map is surjective.

To show (a) and (b), we investigate the diagrams of elements in $N_\mu$. Suppose $v \in N_\mu$. Let

$v_i$ be the entry above the $i$th vertex of $v$,

$v(i)$ be the number of the bottom vertex connected to the $i$th top vertex, so that $\{v_1, v_2, \ldots, v_n\}$ are the labels above the vertices of $v$ and $(v(1), v(2), \ldots, v(n))$ is the permutation determined the diagram (and ignoring the labels $v_i$). By (4.8),

$$\psi_\mu(x_{ij}(t)) = \begin{cases} \psi(t), & \text{if } j = i+1 \text{ and } \mu_{(i)} = 1, \\ 1, & \text{otherwise.} \end{cases} \tag{A}$$

Recall that $v \in N_\mu$ if and only if $u, vuv^{-1} \in U$ implies $\psi_\mu(u) = \psi_\mu(vuv^{-1})$. That is, $v \in N_\mu$ if and only if for all $1 \le i < j \le n$ such that $v(i) < v(j)$,

$$\begin{aligned} \psi_\mu(x_{ij}(t)) &= \psi_\mu(vx_{ij}(t)v^{-1}) \\ &= \psi_\mu(x_{v(i)v(j)}(v_i t v_j^{-1})) \\ &= \begin{cases} \psi(v_i t v_j^{-1}), & \text{if } v(j) = v(i)+1 \text{ and } \mu_{(v(i))} = 1, \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{B}$$

Compare (A) and (B) to obtain that $v \in N_\mu$ if and only if for all $1 \le i < j \le n$ such that $v(i) < v(j)$,

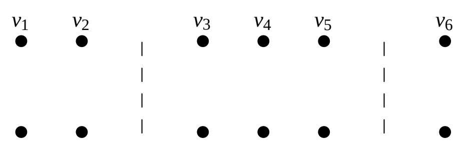
(i) If $\mu_{(i)} = 1$ and $\mu_{(v(i))} = 0$, then $j \neq i+1$,

(ii) If $\mu_{(i)} = 0$ and $\mu_{(v(i))} = 1$, then $v(j) \neq v(i)+1$,

(iii) If $\mu_{(i)} = \mu_{(v(i))} = 1$, then $j = i+1$ if and only if $v(j) = v(i)+1$,

(iii)′ If $\mu_{(i)} = \mu_{(v(i))} = 1$ and $v(j) = v(i)+1$, then $v_i = v_{i+1}$.

We can visualize the implications of the conditions (i)–(iii)′ in the following way. Partition the vertices of $v \in N_\mu$ by $\mu$. For example, $\mu = (2, 3, 1)$ partitions $v$ according to

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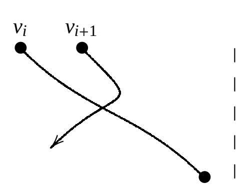
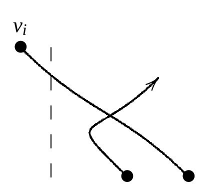
Suppose the $i$th top vertex is not next to a dotted line and the $v(i)$th bottom vertex is immediately to the left of a dotted line. Then condition (i) implies that $v(i+1) < v(i)$, so



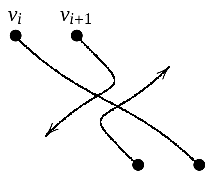

(I)

Similarly, condition (ii) implies

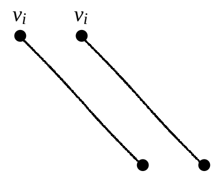



(II)

and conditions (iii) and (iii)′ imply

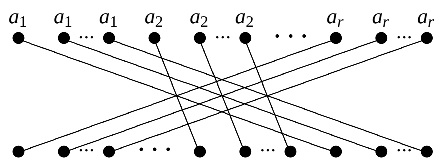



or




$(v_{i+1} = v_i$ in the second). (III)

In the case $\mu = (n)$ condition (III) implies that every $v \in N_{(n)}$ is of the form




$$= w_{(n)}(a_1 w_{(i_1)} \oplus a_2 w_{(i_2)} \oplus \cdots \oplus a_r w_{(i_r)}),$$

where $(i_1, i_2, \ldots, i_r) \models n$, $a_i \in \mathbb{F}_q^*$, and $w_{(k)} \in W_k$ is as in (4.13). In fact, this observation proves that the map $(f) \mapsto v_{(f)}$ is a bijection between $M_{(n)}$ and $N_{(n)}$ (mentioned in the remark).

Note that since the diagrams $v_a$ satisfy (I), (II) and (III), $v_a \in N_\mu$, proving (a). On the other hand, (I), (II) and (III) imply that each $v \in N_\mu$ must be of the form $v = v_a$ for some $a \in M_\mu$, proving (b). □

**Remark:** This bijection will prove useful in developing a generalization of the RSK correspondence in Chapter 5.

Let

$$m_\mu = \{a \in M_\ell(\mathbb{Z}_{\geq 0}) \mid \text{row-sums and column-sums are } \mu\}$$

and for $a \in m_\mu$, let

$$\ell(a) = \text{Card}\{a_{ij} \neq 0 \mid 1 \leq i, j \leq \ell\}$$

**Corollary 4.3.** *Let $\mu \models n$. Then*

$$\dim(\mathcal{H}_\mu) = \sum_{a \in m_\mu} (q-1)^{\ell(a)} q^{n-\ell(a)}$$

*Proof.* Theorem 4.2 gives

$$\dim(\mathcal{H}_\mu) = |M_\mu|.$$

We can obtain a matrix $\bar{a} \in M_\mu$ from a matrix $a \in m_\mu$ by selecting for each $1 \leq i, j \leq \ell(\mu)$, a monic polynomial $f_{ij}$ with nonzero constant term and degree $a_{ij}$. Conversely, every $\bar{a} \in M_\mu$ arises uniquely out of such a construction.

For a fixed $a \in m_\mu$, the total number of ways to choose appropriate polynomials is

$$\prod_{\substack{1 \leq i,j \leq \ell \\ a_{ij} \neq 0}} \text{Card}\{f \in \mathbb{F}_q[X] \mid f \text{ monic, } f(0) = 0 \text{ and } d(f) = a_{ij}\} = (q-1)^{\ell(a)} q^{n-\ell(a)}.$$

The result follows by summing over all $a \in m_\mu$. □

## 4.3 Multiplication in $\mathcal{H}_\mu$

This section examines the implications of the unipotent Hecke algebra multiplication relations from Chapter 3 in the case $G = GL_n(\mathbb{F}_q)$. The final goal is the algorithm given in Theorem 4.5.

### 4.3.1 Pictorial versions of $e_\mu v e_\mu$

Note that the map

$$\begin{aligned} \pi : N = WT \longrightarrow & \quad W \\ wh \mapsto & \quad w, \quad \text{for } w \in W, h \in T, \end{aligned} \tag{4.15}$$

is a surjective group homomorphism. Since $W \subseteq N$, adjust the decomposition of (3.4) as follows. Let $u \in N$ with $\pi(u) = s_{i_1} \cdots s_{i_r}$ for $r$ minimal. Then $u$ has a unique decomposition

$$u = u_1 u_2 \ldots u_r u_T, \qquad \text{where } u_k = s_{i_k},\ u_T \in T. \tag{4.16}$$

For $t \in \mathbb{F}_q$, write $u_k(t) = s_{i_k} x_{i_k}(t)$.

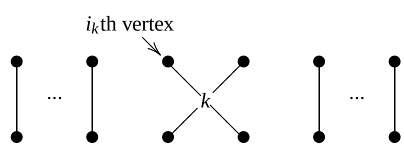
Fix a composition $\mu$. The decomposition

$$U = \prod_{1 \le i < j \le n} U_{ij} \qquad \text{where} \quad U_{ij} = \langle x_{ij}(t) \mid t \in \mathbb{F}_q \rangle,$$

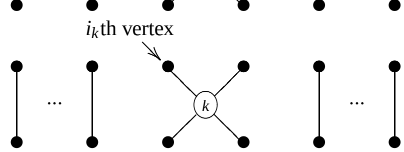
implies

$$e_\mu = \prod_{1 \le i < j \le n} e_{ij}(1), \qquad \text{where} \quad e_{ij}(k) = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(-\mu_{ij} k t) x_{ij}(t). \tag{4.17}$$

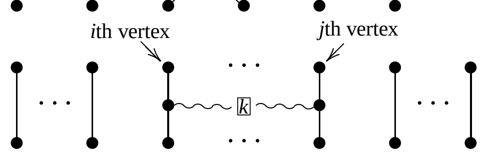
Pictorially, let

$u_k$ as 

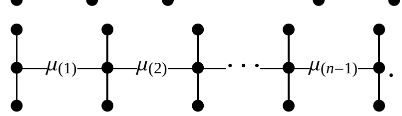
(4.18)

$u_k(t_k)$ as 

(4.19)

$e_{ij}(k)$ as 

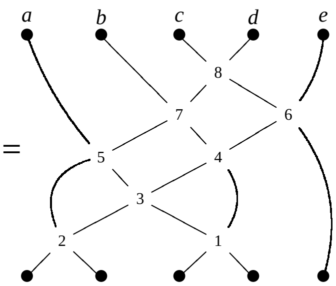
, (4.20)

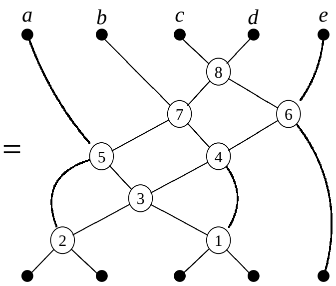
$e_\mu$ as 

. (4.21)

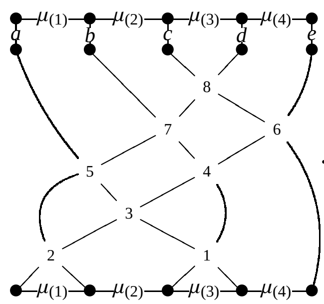
**Examples:**

1. If $u = u_1 u_2 \cdots u_8 u_T \in N$ decomposes according to $s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in W$ with $u_T = \mathrm{diag}(a, b, c, d, e)$, then

$u =$ 

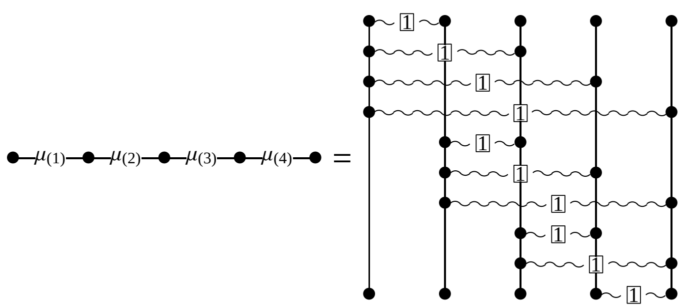
, $u_1(t_1)u_2(t_2)\cdots u_8(t_8)u_T =$ 


and

$e_\mu u e_\mu =$ 

.

2. If $n = 5$, then (4.17) implies




$(e_\mu = e_{45}(1)e_{35}(1)e_{34}(1)e_{25}(1)e_{24}(1)e_{23}(1)e_{15}(1)e_{14}(1)e_{13}(1)e_{12}(1))$.

The elements $e_{ij}(k)$ also interact with $U$ and $N$ as follows (see also Section 3.3.1)

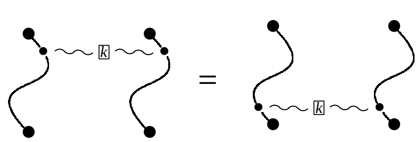
$$s_r e_{ij}(k) s_r = e_{s_r(i)s_r(j)}(k), \tag{E1}$$

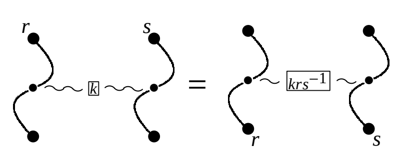
$$e_\mu v e_{ij}(1) = e_\mu v, \qquad v \in N_\mu, (\pi v)(i) < (\pi v)(j), \tag{E2}$$

$$e_{ij}(k) h_{\varepsilon_r}(t) = h_{\varepsilon_r}(t) e_{ij}(t^{\delta_{ri}} t^{-\delta_{rj}} k), \tag{E3}$$

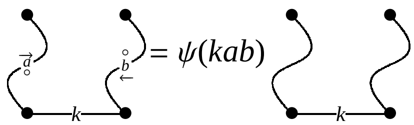
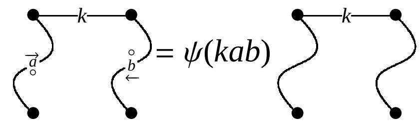
$$e_\mu x_{ij}(t) = \psi(\mu_{ij} t) e_\mu = x_{ij}(t) e_\mu, \tag{E4}$$

or pictorially,

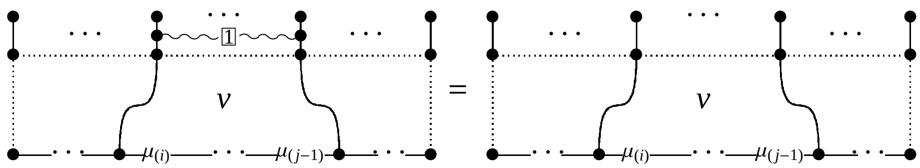


$(E1)$


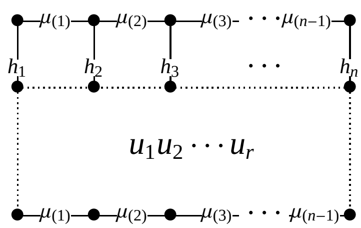
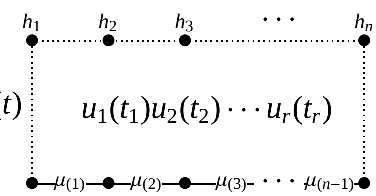
$(E3)$

 and  (E4)

and for $v \in N_\mu$,

 (E2)

Suppose $u = u_1 u_2 \cdots u_r u_T \in N_\mu$ with $u_T = \mathrm{diag}(h_1, h_2, \ldots, h_n)$. Then using (4.17), (E3), (E1) and (E2), we can rewrite $e_\mu u e_\mu$ as

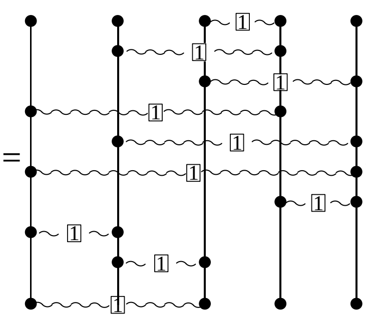
 $= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t)$  (4.22)

where $f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r]$ is given by

$$f_u(y_1, y_2, \ldots, y_r) = -\mu_{i_1 j_1} h_{i_1} h_{j_1}^{-1} y_1 - \mu_{i_2 j_2} h_{i_2} h_{j_2}^{-1} y_2 - \cdots - \mu_{i_r j_r} h_{i_r} h_{j_r}^{-1} y_r, \tag{4.23}$$

for $(i_k, j_k) = (a, b)$, if the $k$th crossing in $u$ crosses the strands coming from the $a$th and $b$th top vertices.

**Example (continued)**. Suppose $u = u_1 u_2 \cdots u_8 u_T \in N$ decomposes according to $s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in W$ and $u_T = \mathrm{diag}(a, b, c, d, e) \in T$ (as in Example 1 above). Consider the ordering

$e_\mu =$ , or $e_\mu = e_{13}(1)e_{23}(1)e_{12}(1)e_{45}(1)e_{15}(1)e_{25}(1)e_{14}(1)e_{35}(1)e_{24}(1)e_{34}(1)$. Then

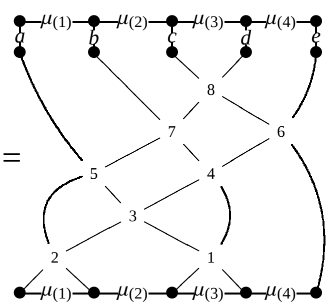
$$\begin{aligned}
e_\mu u e_\mu &= e_\mu u \overleftarrow{e_{13}(1)e_{23}(1)} e_{12}(1)e_{45}(1)e_{15}(1)e_{25}(1)e_{14}(1)e_{35}(1)e_{24}(1)e_{34}(1) \\
&= e_\mu u e_{12}(1)e_{45}(1)e_{15}(1)e_{25}(1)e_{14}(1)e_{35}(1)e_{24}(1)e_{34}(1) && \text{(by (E2))} \\
&= e_\mu u_1 u_2 \dots u_8 u_T \overleftarrow{e_{12}(1)e_{45}(1)e_{15}(1)e_{25}(1)e_{14}(1)e_{35}(1)e_{24}(1)e_{34}(1)} \\
&= e_\mu u_1 u_2 \dots u_8 \overleftarrow{e_{12}(\tfrac{a}{b})e_{45}(\tfrac{d}{e})e_{15}(\tfrac{a}{e})e_{25}(\tfrac{b}{e})e_{14}(\tfrac{a}{d})e_{35}(\tfrac{c}{e})e_{24}(\tfrac{b}{d})e_{34}(\tfrac{c}{d})u_T} && \text{(by (E3))} \\
&= e_\mu u_1 e_{34}(\tfrac{a}{b})u_2 e_{12}(\tfrac{d}{e})u_3 e_{23}(\tfrac{a}{e})u_4 e_{34}(\tfrac{b}{e})u_5 e_{12}(\tfrac{a}{d})u_6 e_{45}(\tfrac{c}{e})u_7 e_{23}(\tfrac{b}{d})u_8 e_{34}(\tfrac{c}{d})u_T && \text{(by (E1))} \\
&= \frac{e_\mu}{q^8} \sum_{t \in \mathbb{F}_q^8} (\psi \circ f)(t) u_1 x_{34}(\tfrac{a}{b}t_1) u_2 x_{12}(\tfrac{d}{e}t_2) u_3 x_{23}(\tfrac{a}{e}t_3) \cdots u_8 x_{34}(\tfrac{c}{d}t_8) u_T,
\end{aligned}$$

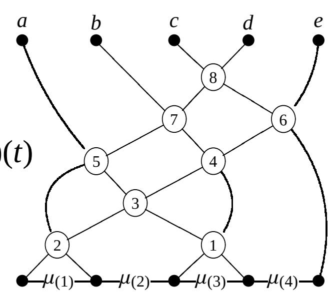
where by (4.17) , $f = -y_1 - y_2 - y_8$. Therefore, by renormalizing

$$\begin{aligned}
e_\mu u e_\mu &= \frac{e_\mu}{q^8} \sum_{t' \in \mathbb{F}_q^8} (\psi \circ f_u)(t') u_1 x_{34}(t_1) u_2 x_{12}(t_2) u_3 x_{23}(t_3) \cdots u_8 x_{34}(t_8) u_T \\
&= \frac{e_\mu}{q^8} \sum_{t' \in \mathbb{F}_q^8} (\psi \circ f_u)(t') u_1(t_1) u_2(t_2) u_3(t_3) \cdots u_8(t_8) u_T,
\end{aligned}$$

where $f_u = -ba^{-1}y_1 - ed^{-1}y_2 - dc^{-1}y_8$. Pictorially, by sliding the $e_{ij}(1)$ down along the strands until they get stuck, this computation gives

$$e_\mu u e_\mu = \quad \text{[diagram]} \quad = \frac{1}{q^8} \sum_{t \in \mathbb{F}_q^8} (\psi \circ f_u)(t) \quad \text{[diagram]} \qquad (4.24)$$



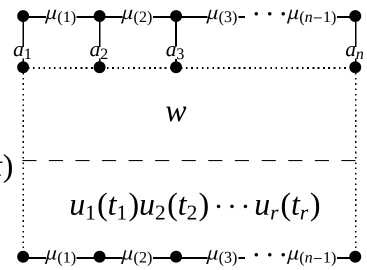

where $f_u = -ba^{-1}y_1 - ed^{-1}y_2 - dc^{-1}y_8$ (as in (4.23)), since $(i_1, j_1) = (1, 2)$, $(i_2, j_2) = (4, 5)$, $(i_3, j_3) = (1, 5)$, etc.

### 4.3.2 Relations for multiplying basis elements.

Let $u = u_1 u_2 \cdots u_r u_T \in N_\mu$ decompose according to a minimal expression in $W$ as in (4.16). Let $v \in N_\mu$ and use (N3) and (N4) to write $u_T v = w \cdot \mathrm{diag}(a_1, a_2, \cdots, a_n)$ for some

$w = \pi(v) \in W$ (see (4.7)). Then use (4.22) to write

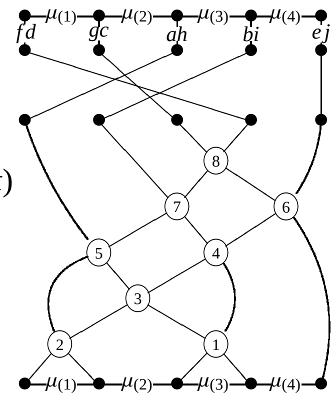
$$(e_\mu u e_\mu)(e_\mu v e_\mu) = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t) \qquad (4.25)$$




(This form corresponds to $\Xi^{\emptyset}(u, u_T v)$ of Corollary 3.9).

**Example (continued)**. If $u$ is as in (4.24) and $v = s_2 s_3 s_2 s_1 s_2 \cdot \mathrm{diag}(f, g, h, i, j) \in N$, then

$$(e_\mu u e_\mu v e_\mu) = \frac{1}{q^8} \sum_{t \in \mathbb{F}_q^8} (\psi \circ f_u)(t) \quad .$$



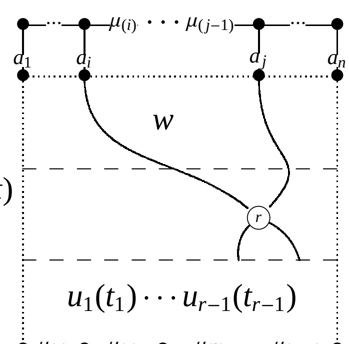

Consider the crossing in (4.25) corresponding to $u_r(t_r)$. There are two possibilities.

**Case 1** the strands that cross at ⓡ do not cross again as they go up to the top of the diagram ($\ell(u_r w) > \ell(w)$),

**Case 2** the strands that cross at ⓡ cross once on the way up to the top of the diagram ($\ell(u_r w) < \ell(w)$).

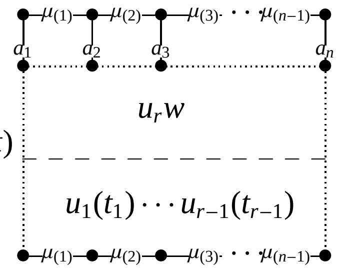
In the first case,

$$e_\mu u e_\mu v e_\mu = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t) \quad ,$$



so (UN1), (UN2) and (E4) imply

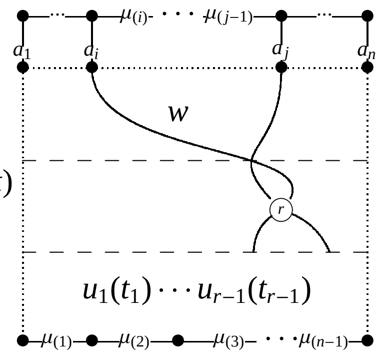
$$e_\mu u e_\mu v e_\mu = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f^{(+0)})(t) \quad \text{(4.26)}$$




where $f^{(+0)} = f_u + \mu_{ij} a_j a_i^{-1} y_r$.

In the second case,

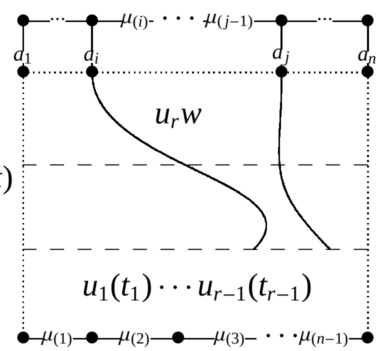
$$e_\mu u e_\mu v e_\mu = \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t)$$



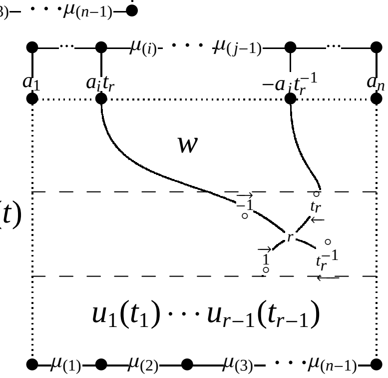

.

Use (UN3) and (N1) to split the sum into two parts corresponding to $t_r = 0$ and $t_r \neq 0$,

$$e_\mu u e_\mu v e_\mu = \frac{1}{q^r} \sum_{\substack{t \in \mathbb{F}_q^r \\ t_r = 0}} (\psi \circ f^{(-0)})(t)$$

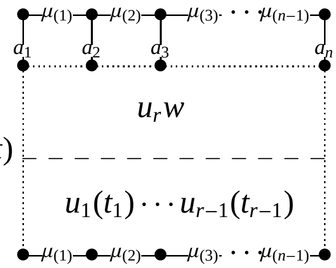



$$+ \frac{1}{q^r} \sum_{\substack{t \in \mathbb{F}_q^r \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f_u)(t)$$

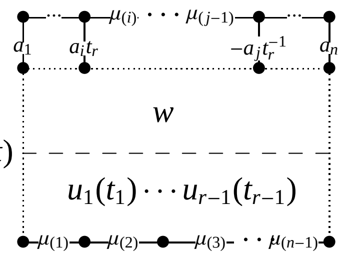


where $f^{(-0)} = f_u$. Now use (UN1), (UN2), (U2), (U1) and (E4) on the second sum to get

$$e_\mu u e_\mu v e_\mu = \frac{1}{q^r} \sum_{\substack{t \in \mathbb{F}_q^r \\ t_r = 0}} (\psi \circ f^{(-0)})(t)$$

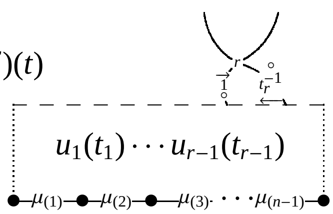



$$+ \frac{1}{q^r} \sum_{\substack{t \in \mathbb{F}_q^r \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f^{(1)})(t) \tag{4.27}$$

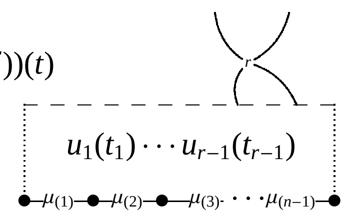



where $f^{(1)} = \varphi_r(f_u) + \mu_{ij} a_j a_i^{-1} y_r^{-1}$, and $\varphi_r(f)$ is defined by

$$\sum_{\substack{t \in \mathbb{F}_q^r \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f)(t)$$




$$= \sum_{\substack{t \in \mathbb{F}_q^r \\ t_r \in \mathbb{F}_q^*}} (\psi \circ \varphi_r(f))(t) \tag{$*$}$$




**Remarks:**

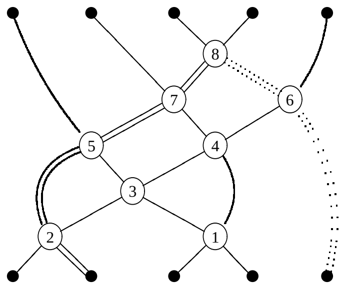
(a) We could have applied these steps for any $f$, $u$, and $v$, so we can iterate the process with each sum.

(b) The most complex step in these computations is determining $\varphi_r$. The following section will develop an efficient algorithm for computing the right-hand side of $(*)$.

### 4.3.3 Computing $\varphi_k$ via painting, paths and sinks.

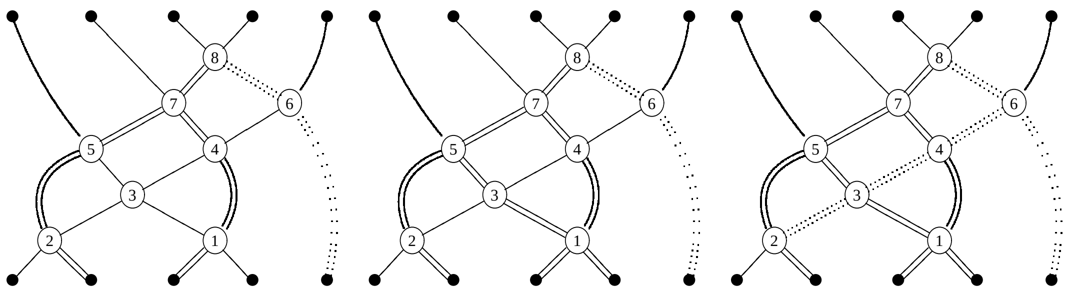
**Painting algorithm ($u^{ⓚ}$).** Suppose $u = u_1 u_2 \cdots u_r \in N$ decomposes according to $s_{i_1} s_{i_2} \cdots s_{i_k} \in W$ (assume $u_T = 1$). Paint flows down strands (by gravity). Each step is illustrated with the example $u = u_1 u_2 \cdots u_8$, decomposed according to $s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3 \in W$.

(1) Paint the left [respectively right] strand exiting ⓚ red [blue] all the way to the bottom of the diagram.




where red is ═, blue is ⋯, and ⓚ is ⑧.

(2) For each crossing that the red [blue] strand passes through, paint the right [left] strand (if possible) red [blue] until that strand either reaches the bottom or crosses the blue [red] strand of (1).

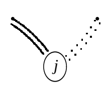


(4.28)

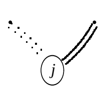
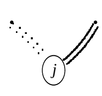
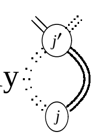
Let

$$u^{ⓚ} = u_1(t_1)u_2(t_2)\cdots u_k(t_k) \text{ painted according to the above algorithm} \tag{4.29}$$

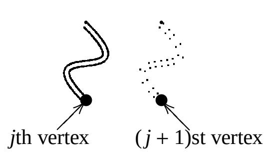
**Sinks and paths**. The diagram $u^{ⓚ}$ has a *crossed sink at* ⓙ if ⓙ is a crossing between a red strand and a blue one, or



Note that since $u$ is decomposed according to a minimal expression in $W$, there will be no crossings of the form

 (since  would imply  for some $j' \geq j$.)

The diagram $u^{ⓚ}$ has a *bottom sink at* $j$ if a red strand enters $j$th bottom vertex *and* a blue strand enters the $(j+1)$st bottom vertex, or



A *red [respectively blue] path* $p$ from a sink $s$ (either crossed or bottom) in $u^{\textcircled{k}}$ is an increasing sequence

$$j_1 < j_2 < \cdots < j_l = k,$$

such that in $u^{\textcircled{k}}$

(a) $\textcircled{j_m}$ is directly connected (no intervening crossings) to $\textcircled{j_{m+1}}$ by a red [blue] strand,

(b) if $s$ is a crossed sink, then $\textcircled{j_1} = s$,

(b′) if $s$ is a bottom sink, then

- in a red path, the $s$th bottom vertex connects to the crossing $\textcircled{j_1}$ with a red strand.
- in a blue path, the $(s+1)$st bottom vertex connects to the crossing $\textcircled{j_1}$ with a blue strand.

Let

$$P_{=}(u^{\textcircled{k}}, s) = \left\{ \begin{matrix} \text{red paths from} \\ s \text{ in } u^{\textcircled{k}} \end{matrix} \right\} \quad \text{and} \quad P_{:}(u^{\textcircled{k}}, s) = \left\{ \begin{matrix} \text{blue paths from} \\ s \text{ in } u^{\textcircled{k}} \end{matrix} \right\} \tag{4.30}$$

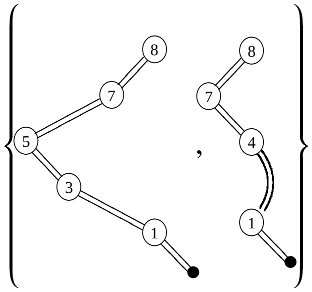
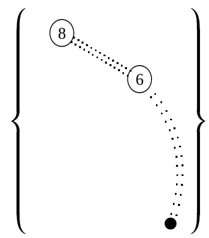
The *weight* of a path $p$ is

$$\mathrm{wt}(p) = \begin{cases} \displaystyle\prod_{\substack{p \text{ switches} \\ \text{strands at } \textcircled{i}}} y_i, & \text{if } p \in P_{=}(u^{\textcircled{k}}, s), \\ \displaystyle\prod_{\substack{p \text{ switches} \\ \text{strands at } \textcircled{i}}} (-y_i), & \text{if } p \in P_{:}(u^{\textcircled{k}}, s). \end{cases} \tag{4.31}$$

Each sink $s$ in $u^{\textcircled{k}}$ (either crossed $\textcircled{j}$ or bottom $j$) has an associated polynomial $g_s \in \mathbb{F}_q[y_1, y_2, \ldots, y_{k-1}, y_k^{-1}]$ given by

$$g_s = \sum_{\substack{p \in P_{=}(u^{\textcircled{k}}, s) \\ p' \in P_{:}(u^{\textcircled{k}}, s)}} \mathrm{wt}(p) y_k^{-1} \mathrm{wt}(p'). \tag{4.32}$$

**Example (continued).** If $u = u_1 u_2 \cdots u_8$ decomposes according to $s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3$ (as in (4.28)), then

$P_{=}(u^{\textcircled{8}}, 4) =$  $P_{:}(u^{\textcircled{8}}, 4) =$ 

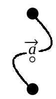
with wt$(1 < 3 < 5 < 7 < 8) = y_5$, wt$(1 < 4 < 7 < 8) = y_1 y_7$, and wt$(6 < 8) = 1$. The corresponding polynomial is

$$g_4 = y_5 y_8^{-1} + y_1 y_7 y_8^{-1}. \tag{4.33}$$

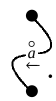
**Lemma 4.4.** *Let $u = u_1 u_2 \cdots u_r$ and $\varphi_r$ be as in (4.27) and $(*)$; suppose $u^{ⓡ}$ is painted as above. Then*

$$\varphi_r(f) = f\Big|_{\{y_j \mapsto y_j - g_{ⓙ} \mid ⓙ \text{ a crossed sink}\}} + \sum_{\substack{j \text{ a bottom} \\ \text{sink}}} \mu_{(j)} g_j.$$

*Proof.* In the painting,

 marks a strand travelled by 

and

 marks a strand travelled by .

Substitutions due to crossed sinks correspond to the normalizations in relation (U2), and the sum over bottom sinks comes from applications of relation (E4). ◻

**Example (continued)** Recall $u = s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3$. Then $u^{⑧}$ has crossed sinks at ②, ③, and ④. The only bottom sink is at 4. Therefore,

$$\varphi_8(f) = f\Big|_{\substack{y_4 \mapsto y_4 - g_{④} \\ y_3 \mapsto y_3 - g_{③} \\ y_2 \mapsto y_2 - g_{②}}} + \mu_{(4)} g_4 = f\Big|_{\substack{y_4 \mapsto y_4 + y_7 y_8^{-1} y_6 \\ y_3 \mapsto y_3 + y_5 y_8^{-1} y_6 \\ y_2 \mapsto y_2 + y_8^{-1} y_6}} + \mu_{(4)}(y_5 y_8^{-1} + y_1 y_7 y_8^{-1}).$$

(for example, $g_4$ was computed in (4.33)).

### 4.3.4 A multiplication algorithm

**Theorem 4.5 (The algorithm).** *Let $G = GL_n(\mathbb{F}_q)$ and $u, v \in N_\mu$. An algorithm for multiplying $e_\mu u e_\mu$ and $e_\mu v e_\mu$ is*

(1) *Decompose $u = u_1 u_2 \cdots u_r u_T$ according to some minimal expression in $W$ (as in (4.16)).*

(2) *Put $e_\mu u e_\mu v e_\mu$ into the form specified by (4.25), with $u_T v = w \cdot \mathrm{diag}(a_1, a_2, \ldots, a_n)$ ($w = \pi(v) \in W$).*

(3) *Complete the following*

   (a) *If $\ell(u_r w) > \ell(w)$, then apply relation (4.26).*

*(b) If $\ell(u_r w) < \ell(w)$, then apply relation (4.27), using $(u_1u_2\cdots u_r)^{ⓡ}$ and Lemma 4.4 to compute $\varphi_r$.*

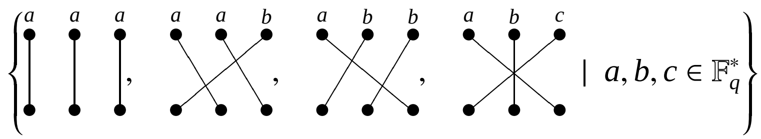
*(4) If $r > 1$, then reapply (3) to each sum with $r := r-1$ and with*

*(a) $w := u_r w$, after using (3a) or using (3b), in the first sum,*

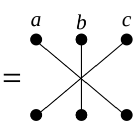
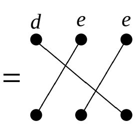
*(b) $w := w$, after using (3b), in the second sum.*

*(5) Set all diagrams not in $N_\mu$ to zero.*

**Sample computation.** Suppose $n = 3$ and $\mu_{(i)} = 1$ for all $1 \le i \le 3$ (i.e.. the Gelfand-Graev case). Then

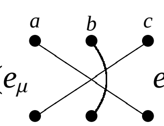
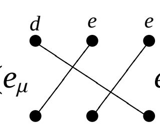
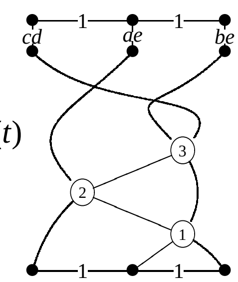
$N_\mu =$ 

Suppose

$u =$  and $v =$ .

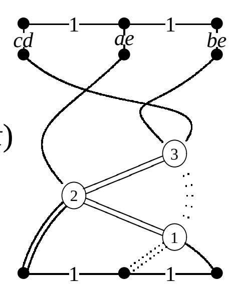
**1. Theorem 4.5 (1):** Let $u = u_1u_2u_3u_T \in N_\mu$ decompose according to $s_2s_1s_2 \in W$, with $u_T = \mathrm{diag}(a,b,c)$.

**2. Theorem 4.5 (2):** By (4.25)

$(e_\mu$  $e_\mu)(e_\mu$  $e_\mu) = \frac{1}{q^3}\sum_{t\in\mathbb{F}_q^3}(\psi\circ f_u)(t)$ 

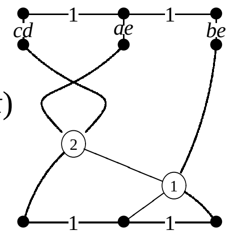
with $u_Tv = s_2s_1\cdot\mathrm{diag}(cd, ae, be)$ (so $w = s_2s_1$), and $f_u = -\frac{b}{a}y_1 - \frac{c}{b}y_3$ (as in (4.23)).

**3. Theorem 4.5 (3b):** Since $\ell(u_3w) < \ell(w)$, paint $u_1(t_1)u_2(t_2)u_3(t_3)$ to get $(u_1u_2u_3)^{③}$ (as in (4.29)),

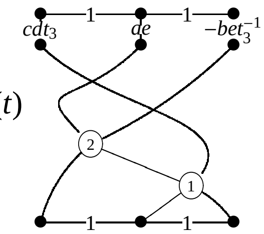
$= \frac{1}{q^3}\sum_{t\in\mathbb{F}_q^3}(\psi\circ f_u)(t)$ 

Now apply (4.27),

$$= \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_3 = 0}} (\psi \circ f^{(-0)})(t)$$




$$+ \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_3 \in \mathbb{F}_q^*}} (\psi \circ f^{(1)})(t)$$



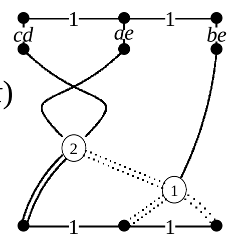

where $f^{(-0)} = -\frac{b}{a}y_1 - \frac{c}{b}y_3$ and by Lemma 4.4,

$$f^{(1)} = \varphi_3(f_u) + \mu_{13}\frac{be}{cd}y_3^{-1} = -\frac{b}{a}y_1 + \frac{b}{a}y_2y_3^{-1} - \frac{c}{b}y_3 + y_3^{-1}.$$

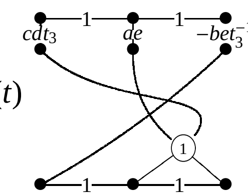
**4. Theorem 4.5 (4):** Set $r := 2$ with $w := u_r w = s_1$ in the first sum and $w := w$ in the second sum.

**5. Theorem 4.5 (3a) (3b):** In the first sum, $\ell(u_2 s_1) < \ell(s_1)$, so paint $u_1(t_1)u_2(t_2)$ to get $(u_1u_2)^{②}$. In the second sum, $\ell(u_2s_2s_1) > \ell(s_2s_1)$, so apply (4.26),

$$= \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_3 = 0}} (\psi \circ f^{(-0)})(t)$$

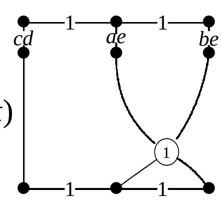



$$+ \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_3 \in \mathbb{F}_q^*}} (\psi \circ f^{(+0,1)})(t)$$

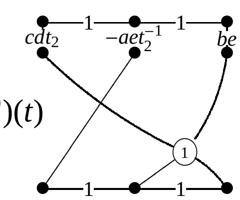



where $f^{(+0,1)} = -\frac{b}{a}y_1 + \frac{b}{a}y_2y_3^{-1} - \frac{c}{b}y_3 + y_3^{-1} - \mu_{(3)}\frac{b}{a}y_2y_3^{-1} = -\frac{b}{a}y_1 - \frac{c}{b}y_3 + y_3^{-1}$. Now apply (4.27) to the first sum,

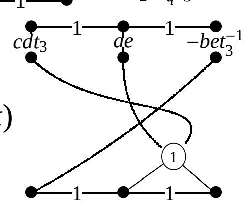
$$= \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_2 = t_3 = 0}} (\psi \circ f^{(-0,-0)})(t)$$




$$+ \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_2 \in \mathbb{F}_q^*, t_3 = 0}} (\psi \circ f^{(1,-0)})(t)$$




$$+ \frac{1}{q^3} \sum_{\substack{t \in \mathbb{F}_q^3 \\ t_3 \in \mathbb{F}_q^*}} (\psi \circ f^{(+0,1)})(t)$$

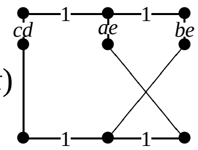



where $f^{(-0,-0)} = -\frac{b}{a}y_1 - \frac{c}{b}y_3$ and $f^{(1,-0)} = \varphi(f^{(-0)}) + \mu_{(1)}\frac{ae}{cd}y_2^{-1} = -\frac{b}{a}y_1 - y_2^{-1}y_1 + \frac{ae}{cd}y_2^{-1}$.

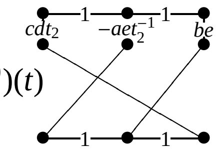
**6. Theorem 4.5 (4):** Set $r = 1$ with $w := u_2 s_1 = 1$ in the first sum, $w := s_1$ in the second sum, and $w := s_1s_2s_1$ in the third sum.

**7. Theorem 4.5 (3a) (3a) (3b):** In the first sum $\ell(s_2 1) > \ell(1)$, so apply (4.26); in the second sum $\ell(s_2 s_1) > \ell(s_1)$, so apply (4.26); in the third sum, $\ell(s_2 s_1 s_2 s_1) < \ell(s_1 s_2 s_1)$, so paint $u_1(t_1)$ to get $u_1^{①}$,

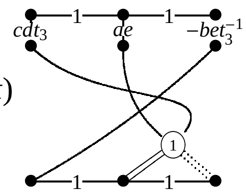
$$= \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2=t_3=0}}(\psi\circ f^{(+0,-0,-0)})(t)$$




$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2\in\mathbb{F}_q^*,t_3=0}}(\psi\circ f^{(+0,1,-0)})(t)$$

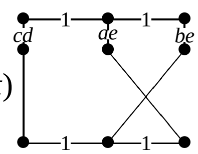



$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_3\in\mathbb{F}_q^*}}(\psi\circ f^{(+0,1)})(t)$$

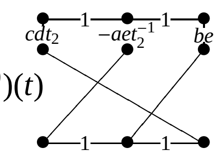



where $f^{(+0,-0,-0)} = -\frac{b}{a}y_1 - \frac{c}{b}y_3 + \frac{b}{a}y_1 = -\frac{c}{b}y_3$ and $f^{(+0,1,-0)} = -\frac{b}{a}y_1 + \frac{ae}{cd}y_2^{-1} - y_2^{-1}y_1$. Now apply (4.27) to the third sum

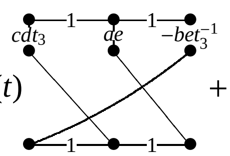
$$= \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2=t_3=0}}(\psi\circ f^{(+0,-0,-0)})(t)$$



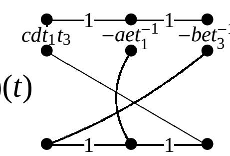

$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_2\in\mathbb{F}_q^*,t_3=0}}(\psi\circ f^{(+0,1,-0)})(t)$$




$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_1=0,t_3\in\mathbb{F}_q^*}}(\psi\circ f^{(-0,+0,1)})(t)$$



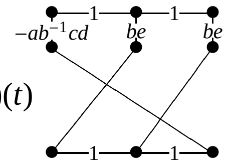

$$+ \frac{1}{q^3}\sum_{\substack{t\in\mathbb{F}_q^3\\ t_1,t_3\in\mathbb{F}_q^*}}(\psi\circ f^{(1,+0,1)})(t)$$

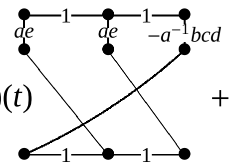


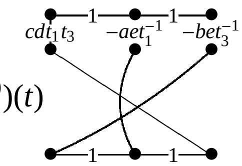

where $f^{(-0,+0,1)} = -\frac{c}{b}y_3 + y_3^{-1}$ and

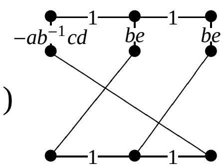
$$f^{(1,+0,1)} = \varphi_1(f^{(+0,1)}) + \mu_{(1)}\frac{ae}{cd}y_1^{-1}y_3^{-1} = -\frac{b}{a}y_1 - \frac{c}{b}y_3 + y_3^{-1} + y_1^{-1} + \frac{ae}{cd}y_1^{-1}y_3^{-1}.$$

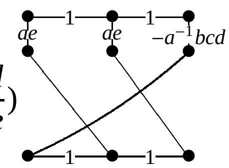
**8. Theorem 4.5 (5):** The first sum contains no elements of $N_\mu$, so set it to zero. The second sum contains elements of $N_\mu$ when $be = -aet_2^{-1}$, so set $t_2 = -\frac{a}{b}$. The third sum contains elements of $N_\mu$ when $cdt_3 = ae$, so set $t_3 = \frac{ae}{cd}$. All the terms in the fourth sum are basis elements.

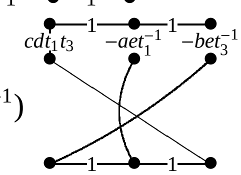
$$= 0 + \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_2=-\frac{a}{b}, t_3=0}} (\psi \circ f^{(+0,1,-0)})(t)$$ 


$$+ \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_1=0, t_3=\frac{ae}{cd}}} (\psi \circ f^{(-0,+0,1)})(t)$$ 

$$+ \frac{1}{q^3} \sum_{\substack{t\in\mathbb{F}_q^3 \\ t_1,t_3\in\mathbb{F}_q^*}} (\psi \circ f^{(1,+0,1)})(t)$$ 


$$= \frac{1}{q^2}\psi(-\frac{be}{cd})$$ 

$$+ \frac{1}{q^2}\psi(-\frac{ae}{bd} + \frac{cd}{ae})$$ 


$$+ \frac{1}{q^2} \sum_{t_1,t_3\in\mathbb{F}_q^*} \psi(-\frac{b}{a}t_1 - \frac{c}{b}t_3 + t_3^{-1} + t_1^{-1} + \frac{ae}{cd}t_1^{-1}t_3^{-1})$$ 

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# Chapter 5

# Representation theory in the $G = GL_n(\mathbb{F}_q)$ case

This chapter examines the representation theory of unipotent Hecke algebras when $G = GL_n(\mathbb{F}_q)$. The combinatorics associated with the representation theory of $GL_n(\mathbb{F}_q)$ generalizes the tableaux combinatorics of the symmetric group, and one of the main results of this chapter is to also give a generalization of the RSK correspondence (see Section 2.3.4).

Fix a homomorphism $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$. Recall that for any composition $\mu \models n$, $\mathcal{H}_\mu$ is a unipotent Hecke algebra (see Chapter 4). A fundamental result concerning Gelfand-Graev Hecke algebras is

**Theorem 5.1 ([GG62],[Yok68],[Ste67]).** *For all $n > 0$, $\mathcal{H}_{(n)}$ is commutative.*

and it will follow from Theorem 5.7, via the representation theory of unipotent Hecke algebras.

## 5.1 The representation theory of $\mathcal{H}_\mu$

Let $\mathcal{S}$ be a set. An *$\mathcal{S}$-partition* $\lambda = (\lambda^{(s_1)}, \lambda^{(s_2)}, \ldots)$ is a sequence of partitions indexed by the elements of $\mathcal{S}$. Let

$$\mathcal{P}^{\mathcal{S}} = \{\mathcal{S}\text{-partitions}\}. \tag{5.1}$$

The following discussion defines two sets $\Theta$ and $\Phi$, so that $\Theta$-partitions index the irreducible characters of $G$ and $\Phi$-partitions index the conjugacy classes of $G$.

Let $L_n = \mathrm{Hom}(\mathbb{F}_{q^n}^*, \mathbb{C}^*)$ be the character group of $\mathbb{F}_{q^n}^*$. If $\gamma \in L_m$, then let

$$\begin{array}{rcl} \gamma_{(r)} : \mathbb{F}_{q^{mr}}^* & \longrightarrow & \mathbb{C}^* \\ x & \mapsto & \gamma(x^{1+q^r+q^{2r}+\cdots+q^{m(r-1)}}) \end{array}$$

Thus if $n = mr$, then we may view $L_m \subseteq L_n$ by identifying $\gamma \in L_m$ with $\gamma_{(r)} \in L_n$. Define

$$L = \bigcup_{n \geq 0} L_n.$$

The *Frobenius maps* are

$$\begin{array}{rcl} F : \overline{\mathbb{F}}_q & \to & \overline{\mathbb{F}}_q \\ x & \mapsto & x^q \end{array} \qquad \text{and} \qquad \begin{array}{rcl} F : L & \to & L \\ \gamma & \mapsto & \gamma^q \end{array},$$

where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$.

The map

$$\begin{array}{ccc} \{\text{F-orbits of } \overline{\mathbb{F}}_q^*\} & \longrightarrow & \{f \in \mathbb{F}_q[t] \mid f \text{ is monic, irreducible, and } f(0) \neq 0\} \\ \{x, x^q, x^{q^2}, \ldots, x^{q^{k-1}}\} & \mapsto & f_x = \prod_{i=1}^{k-1} (t - x^{q^i}), \qquad \text{where } x^{q^k} = x \in \overline{\mathbb{F}}_q^* \end{array}$$

is a bijection such that the size of the $F$-orbit of $x$ equals the degree $d(f_x)$ of $f_x$. Let

$$\Phi = \left\{ f \in \mathbb{C}[t] \;\middle|\; \begin{array}{l} f \text{ is monic, irreducible} \\ \text{and } f(0) \neq 0 \end{array} \right\} \quad \text{and} \quad \Theta = \{F\text{-orbits in } L\}. \tag{5.2}$$

If $\eta$ is a $\Phi$-partition and $\lambda$ is a $\Theta$-partition, then let

$$|\eta| = \sum_{f \in \Phi} d(f)|\eta^{(f)}| \qquad \text{and} \qquad |\lambda| = \sum_{\varphi \in \Theta} |\varphi||\lambda^{(\varphi)}|$$

be the *size* of $\eta$ and $\lambda$, respectively. Let the sets $\mathcal{P}^\Phi$ and $\mathcal{P}^\Theta$ be as in (5.1) and let

$$\mathcal{P}_n^\Phi = \{\eta \in \mathcal{P}^\Phi \mid |\eta| = n\} \qquad \text{and} \qquad \mathcal{P}_n^\Theta = \{\lambda \in \mathcal{P}^\Theta \mid |\lambda| = n\}. \tag{5.3}$$

**Theorem 5.2 (Green [Gre55]).** *Let $G_n = \mathrm{GL}_n(\mathbb{F}_q)$.*

*(a) $\mathcal{P}_n^\Phi$ indexes the conjugacy classes $K^\eta$ of $G_n$,*

*(b) $\mathcal{P}_n^\Theta$ indexes the irreducible $G_n$-modules $G_n^\lambda$.*

Suppose $\lambda \in \mathcal{P}^\Theta$. A *column strict tableau* $P = (P^{(\varphi_1)}, P^{(\varphi_2)}, \ldots)$ *of shape* $\lambda$ is a column strict filling of $\lambda$ by positive integers. That is, $P^{(\varphi)}$ is a column strict tableau of shape $\lambda^{(\varphi)}$. Write $\mathrm{sh}(P) = \lambda$. The *weight of* $P$ is the composition $\mathrm{wt}(P) = (\mathrm{wt}(P)_1, \mathrm{wt}(P)_2, \ldots)$ given by

$$\mathrm{wt}(P)_i = \sum_{\varphi \in \Theta} |\varphi| \begin{pmatrix} \text{number of} \\ i \text{ in } P^{(\varphi)} \end{pmatrix}.$$

If $\lambda \in \mathcal{P}^\Theta$ and $\mu$ is a composition, then let

$$\hat{\mathcal{H}}_\mu^\lambda = \{\text{column strict tableaux } P \mid \mathrm{sh}(P) = \lambda, \mathrm{wt}(P) = \mu\} \tag{5.4}$$

and

$$\hat{\mathcal{H}}_\mu = \{\lambda \in \mathcal{P}^\Theta \mid \hat{\mathcal{H}}_\mu^\lambda \text{ is not empty}\}. \tag{5.5}$$

The following theorem is a consequence of Theorem 2.3 and a theorem proved by Zelevinsky [Zel81] (see Theorem 5.5). A proof of Zelevinsky's theorem is in Section 5.3.

**Theorem 5.3.** *The set $\hat{\mathcal{H}}_\mu$ indexes the irreducible $\mathcal{H}_\mu$-modules $\mathcal{H}_\mu^\lambda$ and*

$$\dim(\mathcal{H}_\mu^\lambda) = |\hat{\mathcal{H}}_\mu^\lambda|.$$

## 5.2 A generalization of the RSK correspondence

For a composition $\mu \models n$, let $N_\mu$ be as in (4.11) and $M_\mu$ as in (4.12).

The $(\mathcal{H}_\mu, \mathcal{H}_\mu)$-bimodule decomposition

$$\mathcal{H}_\mu \cong \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \mathcal{H}_\mu^\lambda \otimes \mathcal{H}_\mu^\lambda \quad \text{implies} \quad |N_\mu| = \dim(\mathcal{H}_\mu) = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} \dim(\mathcal{H}_\mu^\lambda)^2 = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} |\hat{\mathcal{H}}_\mu^\lambda|^2.$$

Theorem 5.4, below, gives a combinatorial proof of this identity.

Encode each matrix $a \in M_\mu$ as a $\Phi$-sequence

$$(a^{(f_1)}, a^{(f_2)}, \ldots), \qquad f_i \in \Phi,$$

where $a^{(f)} \in M_{\ell(\mu)}(\mathbb{Z}_{\geq 0})$ is given by

$$a_{ij}^{(f)} = \text{highest power of } f \text{ dividing } a_{ij}.$$

Note that this is an entry by entry "factorization" of $a$ such that

$$a_{ij} = \prod_{f \in \Phi} f^{a_{ij}^{(f)}}.$$

Recall from Section 2.3.4 the classical RSK correspondence

$$\begin{array}{rcl} M_\ell(\mathbb{Z}_{\geq 0}) & \longrightarrow & \left\{ \begin{array}{l} \text{Pairs } (P,Q) \text{ of column strict} \\ \text{tableaux of the same shape} \end{array} \right\} \\ b & \mapsto & (P(b), Q(b)). \end{array}$$

**Theorem 5.4.** *For $a \in M_\mu$, let $P(a)$ and $Q(a)$ be the $\Phi$-column strict tableaux given by*

$$P(a) = (P(a^{(f_1)}), P(a^{(f_2)}), \ldots) \qquad \text{and} \qquad Q(a) = (Q(a^{(f_1)}), Q(a^{(f_2)}), \ldots) \qquad \text{for } f_i \in \Phi.$$

*Then the map*

$$\begin{array}{ccccc} N_\mu & \longrightarrow & M_\mu & \longrightarrow & \left\{ \begin{array}{l} \textit{Pairs } (P,Q) \textit{ of } \Phi\textit{-column} \\ \textit{strict tableaux of the same} \\ \textit{shape and weight } \mu \end{array} \right\} \\ v & \mapsto & a_v & \mapsto & (P(a_v), Q(a_v)), \end{array}$$

*is a bijection, where the first map is the inverse of the bijection in Theorem 4.2.*

By the construction above, the map is well-defined, and since all the steps are invertible, the map is a bijection.

**Example.** suppose $\mu = (7, 5, 3, 2)$ and $f, g, h \in \Phi$ are such that $d(f) = 1$, $d(g) = 2$, and $d(h) = 3$. Then

$$a_v = \begin{pmatrix} g & f^2h & 1 & 1 \\ h & 1 & g & 1 \\ 1 & 1 & f & f^2 \\ g & 1 & 1 & 1 \end{pmatrix} \in M_{\ydiagram{7,5,3,2}}$$

corresponds to the sequence

$$(a_v^{(f_1)}, a_v^{(f_2)}, \ldots) = \left( \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{(f)}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{(g)}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{(h)} \right)$$

and

$$(P(a_v), Q(a_v)) = \left( \begin{ytableau} 2 & 2 & 4 \\ 3 & 4 \end{ytableau}^{(f)}, \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}^{(g)}, \begin{ytableau} 1 & 2 \end{ytableau}^{(h)} \right) \left( \begin{ytableau} 1 & 1 & 3 \\ 3 & 3 \end{ytableau}^{(f)}, \begin{ytableau} 1 & 4 \\ 2 \end{ytableau}^{(g)}, \begin{ytableau} 1 & 2 \end{ytableau}^{(h)} \right).$$

## 5.3 Zelevinsky's decomposition of $\mathrm{Ind}_U^G(\psi_\mu)$

This section proves the theorem

**Theorem 5.5 (Zelevinsky [Zel81]).** *Let $U$ be the subgroup of unipotent upper-triangular matrices of $G = \mathrm{GL}_n(\mathbb{F}_q)$, $\mu \models n$ and $\psi_\mu$ be as in (4.8). Then*

$$\mathrm{Ind}_U^G(\psi_\mu) = \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \mathrm{Card}(\hat{\mathcal{H}}_\mu^\lambda) G^\lambda.$$

Theorem 5.3 follows from this theorem and Theorem 2.3. The proof of Theorem 5.5 is in 3 steps.

(1) Establish the necessary connection between symmetric functions and the representation theory of $G$.

(2) Prove Theorem 5.5 for the case when $\ell(\mu) = 1$.

(3) Generalize (2) to arbitrary $\mu$.

The proof below uses the ideas of Zelevinsky's proof, but explicitly uses symmetric functions to prove the results. Specifically, the following discussion through the proof of Theorem 5.7 corresponds to [Zel81, Sections 9-11] and Theorem 5.5 corresponds to [Zel81, Theorem 12.1].

### 5.3.1 Preliminaries to the proof (1)

Let $\mu \models n$ and $G = G_n$. The group

$$P_\mu = \left\{ \begin{pmatrix} \boxed{g_1} & & & * \\ & \boxed{g_2} & & \\ & & \ddots & \\ 0 & & & \boxed{g_\ell} \end{pmatrix} \mid g_i \in G_{\mu_i} = \mathrm{GL}_{\mu_i}(\mathbb{F}_q) \right\} \tag{5.6}$$

has subgroups

$$L_\mu = G_{\mu_1} \oplus G_{\mu_2} \oplus \cdots \oplus G_{\mu_\ell} \quad \text{and} \quad U_\mu = \left\{ \begin{pmatrix} \boxed{Id_{\mu_1}} & & & * \\ & \boxed{Id_{\mu_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{Id_{\mu_\ell}} \end{pmatrix} \right\}, \tag{5.7}$$

where $Id_k$ is the $k \times k$ identity matrix. Note that $P_\mu = L_\mu U_\mu$ and $P_\mu = N_G(U_\mu)$. The *indflation map* is a composition of the inflation map and the induction map,

$$\begin{array}{rcccccc} \mathrm{Indf}_{L_\mu}^G : & R[L_\mu] & \longrightarrow & R[P_\mu] & \longrightarrow & R[G] \\ & \chi & \mapsto & \mathrm{Inf}_{L_\mu}^{P_\mu}(\chi) & \mapsto & \mathrm{Ind}_{P_\mu}^G(\mathrm{Inf}_{L_\mu}^{P_\mu}(\chi)), \end{array}$$

$$\text{where} \qquad \begin{array}{rccc} \mathrm{Inf}_{L_\mu}^{P_\mu}(\chi) : & P_\mu & \to & \mathbb{C} \\ & lu & \mapsto & \chi(l), \end{array} \qquad \text{for } l \in L_\mu \text{ and } u \in U_\mu.$$

Suppose $\lambda \in \mathcal{P}^\Theta$ and $\eta \in \mathcal{P}^\Phi$ (see (5.3)). Let $\chi^\lambda$ be the irreducible character corresponding to the irreducible $G$-module $G^\lambda$ and let $\kappa^\eta$ be the characteristic function corresponding to the conjugacy class $K^\eta$ (see Theorem 5.2), given by

$$\kappa^\eta(g) = \begin{cases} 1, & \text{if } g \in K^\eta, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{for } g \in G_{|\eta|}.$$

Define

$$R = \bigoplus_{n \geq 0} R[G_n] = \mathbb{C}\text{-span}\{\chi^\lambda \mid \lambda \in \mathcal{P}^\Theta\} = \mathbb{C}\text{-span}\{\kappa^\eta \mid \eta \in \mathcal{P}^\Phi\}.$$

The space $R$ has an inner product defined by

$$\langle \chi^\lambda, \chi^\nu \rangle = \delta_{\lambda\nu},$$

and multiplication

$$\chi^\lambda \circ \chi^\nu = \mathrm{Indf}_{L_{(r,s)}}^{G_{r+s}}(\chi^\lambda \otimes \chi^\nu), \qquad \text{for } \lambda \in \mathcal{P}_r^\Theta, \nu \in \mathcal{P}_s^\Theta. \tag{5.8}$$

For each $\varphi \in \Theta$, let $\{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \ldots\}$ be an infinite set of variables, and let

$$\Lambda_{\mathbb{C}} = \bigotimes_{\varphi \in \Theta} \Lambda_{\mathbb{C}}(Y^{(\varphi)}),$$

where $\Lambda_{\mathbb{C}}(Y^{(\varphi)})$ is the ring of symmetric functions in $\{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \ldots\}$ (see Section 2.3.3). For each $f \in \Phi$, define an additional set of variables $\{X_1^{(f)}, X_2^{(f)}, \ldots\}$ such that the symmetric functions in the $Y$ variables are related to the symmetric functions in the $X$ variables by the transform

$$p_k(Y^{(\varphi)}) = (-1)^{k|\varphi|-1} \sum_{x \in \mathbb{F}^*_{q^{k|\varphi|}}} \xi(x) p_{\frac{k|\varphi|}{d(f_x)}}(X^{(f_x)}), \tag{5.9}$$

where $\xi \in \varphi$, $f_x \in \Phi$ is the irreducible polynomial that has $x$ as a root, and $p_{\frac{a}{b}}(X^{(f)}) = 0$ if $\frac{a}{b} \notin \mathbb{Z}_{\geq 0}$. Then

$$\Lambda_{\mathbb{C}} = \bigotimes_{f \in \Phi} \Lambda_{\mathbb{C}}(X^{(f)}).$$

For $\nu \in \mathcal{P}$, let $s_\nu(Y^{(\varphi)})$ be the Schur function and $P_\nu(X^{(f)}; t)$ be the Hall-Littlewood symmetric function (as in Section 2.3.3). Define

$$s_\lambda = \prod_{\varphi \in \Theta} s_{\lambda^{(\varphi)}}(Y^{(\varphi)}) \qquad \text{and} \qquad P_\eta = q^{-n(\eta)} \prod_{f \in \Phi} P_{\eta^{(f)}}(X^{(f)}; q^{-d(f)}), \tag{5.10}$$

where $n(\eta) = \sum_{f \in \Phi} d(f) n(\eta^{(f)})$ and for a composition $\mu$, $n(\mu) = \sum_{i=1}^{\ell(\mu)} (i-1)\mu_i$. The ring

$$\Lambda_{\mathbb{C}} = \mathbb{C}\text{-span}\{s_\lambda \mid \lambda \in \mathcal{P}^\Theta\} = \mathbb{C}\text{-span}\{P_\eta \mid \eta \in \mathcal{P}^\Phi\}$$

has an inner product given by

$$\langle s_\lambda, s_\nu \rangle = \delta_{\lambda\nu}.$$

**Theorem 5.6 (Green [Gre55],Macdonald [Mac95]).** *The linear map*

$$\begin{array}{rccll} \mathrm{ch}: & R & \longrightarrow & \Lambda_{\mathbb{C}} & \\ & \chi^\lambda & \mapsto & s_\lambda, & \text{for } \lambda \in \mathcal{P}^\Theta \\ & \kappa^\eta & \mapsto & P_\eta, & \text{for } \eta \in \mathcal{P}^\Phi, \end{array}$$

*is an algebra isomorphism that preserves the inner product.*

A *unipotent conjugacy class* $K^\eta$ is a conjugacy class such that $\eta^{(f)} = \emptyset$ unless $f = t-1$. Let

$$\mathcal{U} = \mathbb{C}\text{-span}\{\kappa^\eta \mid \eta^{(f)} = \emptyset, \text{ unless } f = t-1\} \subseteq R$$

be the subalgebra of unipotent class functions. Note that by (5.10) and Theorem 5.6,

$$\mathrm{ch}(\mathcal{U}) = \Lambda_{\mathbb{C}}(X^{(t-1)}).$$

Consider the projection $\pi : R \to \mathcal{U}$ which is an algebra homomorphism given by

$$(\pi\chi^\lambda)(g) = \begin{cases} \chi^\lambda(g), & \text{if } g \in G \text{ is unipotent,} \\ 0, & \text{otherwise,} \end{cases} \qquad \lambda \in \mathcal{P}^\Theta.$$

Then $\tilde{\pi} = \mathrm{ch} \circ \pi \circ \mathrm{ch}^{-1} : \Lambda_{\mathbb{C}} \to \Lambda_{\mathbb{C}}(X^{(t-1)})$ is given by

$$\begin{aligned} \tilde{\pi}(p_k(Y^{(\varphi)})) &= \tilde{\pi}\left( (-1)^{k|\varphi|-1} \sum_{x \in \mathbb{F}^*_{q^{k|\varphi|}}} \xi(x) p_{\frac{k|\varphi|}{d(f_x)}}(X^{(f_x)}) \right) \quad \text{(by (5.9))} \\ &= (-1)^{k|\varphi|-1} \xi(1) p_{\frac{k|\varphi|}{1}}(X^{(t-1)}) + 0 \\ &= (-1)^{k|\varphi|-1} p_{k|\varphi|}(X^{(t-1)}). \end{aligned} \tag{5.11}$$

### 5.3.2 The decomposition of $\mathrm{Ind}_U^G(\psi_{(n)})$ (2)

The representation $\mathrm{Ind}_U^G(\psi_{(n)})$ is the Gelfand-Graev module, and with Theorem 2.3, Theorem 5.7, below, proves that $\mathcal{H}_{(n)}$ is commutative.

**Theorem 5.7.** *Let $U$ be the subgroup of unipotent upper-triangular matrices of $G = \mathrm{GL}_n(\mathbb{F}_q)$. Then*

$$\mathrm{ch}(\mathrm{Ind}_U^G(\psi_{(n)})) = \sum_{\substack{\lambda \in \mathcal{P}_n^\Theta \\ \mathrm{ht}(\lambda)=1}} s_\lambda, \quad \textit{where } \mathrm{ht}(\lambda) = \max\{\ell(\lambda^{(\varphi)}) \mid \varphi \in \Theta\}.$$

*Proof.* Let

$$\begin{array}{rcl} \Psi : R & \longrightarrow & \mathbb{C} \\ \chi^\lambda & \mapsto & \langle \chi^\lambda, \mathrm{Ind}_U^G(\psi_{(n)}) \rangle \end{array} \quad \text{and} \quad \tilde{\Psi} : \Lambda_{\mathbb{C}} \xrightarrow{\mathrm{ch}^{-1}} R \xrightarrow{\Psi} \mathbb{C}. \tag{5.12}$$

For any finite group $H$ and $\gamma, \chi \in R[H]$, let

$$\begin{array}{rrcl} 1_H : & H & \to & C \\ & h & \mapsto & 1 \end{array}, \quad e_H = \frac{1}{|H|} \sum_{h \in H} h, \quad \text{and} \quad \langle \chi, \gamma \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h)\gamma(h^{-1}).$$

The proof is in six steps.

(a) $\tilde{\Psi}(e_k(Y^{(1)})) = \delta_{k1}$, where $1 = 1_{\mathbb{F}_q^*}$,

(b) $\Psi(\chi^\lambda) = \dim(e_{(n)} G^\lambda)$ for $\lambda \in \mathcal{P}^\Theta$,

(c) $\tilde{\Psi}(fg) = \tilde{\Psi}(f)\tilde{\Psi}(g)$ for all $f, g \in \Lambda_{\mathbb{C}}(Y^{(1)})$, where $1 = 1_{\mathbb{F}_q^*}$.

(d) $\Psi \circ \pi = \Psi$,

(e) $\tilde{\Psi}(f(Y^{(\varphi)})) = \tilde{\Psi}(f(Y^{(1)}))$ for all $f \in \Lambda_{\mathbb{C}}(Y^{(\varphi)})$,

(f) $\tilde{\Psi}(s_\lambda) = \delta_{\mathrm{ht}(\lambda)1}$.

(a) An argument similar to the argument in [Mac95, pgs. 285-286] shows that

$$\mathrm{ch}^{-1}(e_k(Y^{(1)})) = 1_{G_k}$$

(see [HR99, Theorem 4.9 (a)] for details). Therefore, by Frobenius reciprocity and the orthogonality of characters,

$$\tilde{\Psi}(e_k(Y^{(1)})) = \langle 1_{G_k}, \mathrm{Ind}_U^G(\psi_{(n)})\rangle = \langle 1_{U_k}, \psi_{(n)}\rangle_{U_k} = \delta_{k1}.$$

(b) Since there exists an idempotent $e$ such that $G^\lambda \cong \mathbb{C}Ge$ and $\mathrm{Ind}_U^G(\psi_{(n)}) \cong \mathbb{C}Ge_{(n)}$, the map

$$\begin{array}{ccc} e_{(n)}\mathbb{C}Ge & \longrightarrow & \mathrm{Hom}_G(G^\lambda, \mathbb{C}Ge_{(n)}) \\ e_{(n)}ge & \mapsto & \begin{array}{rcl} \gamma_g : \mathbb{C}Ge & \to & \mathbb{C}Ge_{(n)} \\ xe & \mapsto & xege_{(n)} \end{array} \end{array}$$

is a vector space isomorphism (using an argument similar to the proof of [CR81, (3.18)]). Thus,

$$\begin{aligned} \Psi(\chi^\lambda) = \langle \chi^\lambda, \mathrm{Ind}_U^G(\psi_{(n)})\rangle &= \dim(\mathrm{Hom}_G(G^\lambda, \mathrm{Ind}_U^G(\psi_{(n)}))) \\ &= \dim(e_{(n)}\mathbb{C}Ge) = \dim(e_{(n)}G^\lambda). \end{aligned}$$

(c) By (a), $\tilde{\Psi}(e_r(Y^{(1)}))\tilde{\Psi}(e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}$. It therefore suffices to show that

$$\tilde{\Psi}(e_r(Y^{(1)})e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}, \qquad (\text{since } \Lambda_{\mathbb{C}}(Y^{(1)}) = \mathbb{C}[e_1(Y^{(1)}), e_2(Y^{(1)}), \ldots]).$$

Suppose $r + s = n$ and let $P = P_{(r,s)}$. Then

$$\tilde{\Psi}(e_r(Y^{(1)})e_s(Y^{(1)})) = \Psi(\mathrm{Ind}_P^{G_n}(1_P)) = \dim(e_{(n)}\mathbb{C}Ge_P).$$

Since $T \subseteq P$, $e_P = e_{(1^n)}e_P$, $G = \bigsqcup_{v\in N} UvU$, and $N = WT$,

$$e_{(n)}\mathbb{C}Ge_P = e_{(n)}\mathbb{C}Ge_{(1^n)}e_P = \mathbb{C}\text{-span}\{e_{(n)}we_{(1^n)}e_P \mid w \in W\}.$$

If there exists $1 \le i \le n$ such that $w(i+1) = w(i) + 1$, then

$$e_{(n)}we_{(1^n)} = e_{(n)}wx_{i,i+1}(t)e_{(1^n)} = e_{(n)}x_{w(i),w(i)+1}(t)we_{(1^n)} = \psi(t)e_{(n)}we_{(1^n)}.$$

Therefore, $e_{(n)}we_{(1^n)} = 0$ unless $w = w_{(n)}$. If $r > 1$ of $s > 1$, then there exists $1 \le i \le n$ such that $x_{i+1,i}(t) \in P_{(r,s)}$, so

$$e_{(n)}w_{(n)}e_P = e_{(n)}w_{(n)}x_{i+1,i}(t)e_P = e_{(n)}x_{n-i,n-i+1}(t)w_{(n)}e_P = \psi(t)e_{(n)}w_{(n)}e_P = 0.$$

In particular,

$$\dim(e_{(n)}\mathbb{C}Ge_P) = 0.$$

If $r = s = 1$, then $P_{(1,1)}$ is upper-triangular, so

$$e_{(2)}w_{(2)}e_P \neq 0$$

and $\dim(e_{(2)}\mathbb{C}Ge_P) = 1$, giving $\tilde{\Psi}(e_r(Y^{(1)})e_s(Y^{(1)})) = \delta_{r1}\delta_{s1}$.

(d) By Frobenius reciprocity,

$$\begin{aligned}\langle\chi^\lambda, \mathrm{Ind}_{U_n}^{G_n}(\psi_{(n)})\rangle &= \langle\mathrm{Res}_{U_n}^{G_n}(\chi^\lambda), \psi_{(n)}\rangle_{U_n}\\ &= \langle\mathrm{Res}_{U_n}^{G_n}(\pi(\chi^\lambda)), \psi_{(n)}\rangle_{U_n} = \langle\pi(\chi^\lambda), \mathrm{Ind}_{U_n}^{G_n}(\psi_{(n)})\rangle,\end{aligned}$$

so $\Psi = \Psi \circ \pi$.

(e) Induct on $n$, using (c) and the identity

$$(-1)^{n-1}p_n(Y^{(1)}) = ne_n(Y^{(1)}) - \sum_{r=1}^{n-1}(-1)^{r-1}p_r(Y^{(1)})e_{n-r}(Y^{(1)}), \qquad \text{[Mac95, I.2.11']}$$

to obtain $\tilde{\Psi}(p_n(Y^{(1)})) = 1$. Note that

$$\begin{aligned}\tilde{\Psi}(p_n(Y^{(\varphi)})) = \tilde{\Psi}(\pi(p_n(Y^{(\varphi)}))) = \tilde{\Psi}((-1)^{|\varphi|n-1}p_{|\varphi|n}(X^{(t-1)})) = \tilde{\Psi}(\pi(p_{|\varphi|k}(Y^{(1)})))\\ = \tilde{\Psi}(p_{|\varphi|k}(Y^{(1)})) = 1 = \tilde{\Psi}(p_k(Y^{(1)})).\end{aligned}$$

Since $\tilde{\Psi}$ is multiplicative on $\Lambda_{\mathbb{C}}(Y^{(1)})$,

$$\tilde{\Psi}(p_\nu(Y^{(\varphi)})) = 1 = \tilde{\Psi}(p_\nu(Y^{(1)})), \qquad \text{for all partitions } \nu.$$

In particular, since $\tilde{\Psi}$ is linear and $\Lambda_{\mathbb{C}}(Y^{(\varphi)}) = \mathbb{C}\text{-span}\{p_\nu(Y^{(\varphi)})\}$,

$$\tilde{\Psi}(f(Y^{(\varphi)})) = \tilde{\Psi}(f(Y^{(1)})), \qquad \text{for all } f \in \Lambda_{\mathbb{C}}(Y^{(\varphi)}).$$

Note that (e) also implies that $\tilde{\Psi}$ is multiplicative on all of $\Lambda_{\mathbb{C}}$.

(f) Note that

$$\tilde{\Psi}(s_\lambda) = \tilde{\Psi}\left(\prod_{\varphi\in\Theta} s_{\lambda(\varphi)}(Y^{(\varphi)})\right) = \tilde{\Psi}\left(\prod_{\varphi\in\Theta} s_{\lambda(\varphi)}(Y^{(1)})\right) = \prod_{\varphi\in\Theta}\tilde{\Psi}(s_{\lambda(\varphi)}(Y^{(1)})),$$

where the last two equalities follow from (e) and (c), respectively. By definition $s_\nu(Y^{(1)}) = \det(e_{\nu'_i-i+j}(Y^{(1)}))$, so

$$\tilde{\Psi}(s_\nu(Y^{(1)})) = \begin{cases} 1, & \text{if } \ell(\nu) = 1,\\ 0, & \text{otherwise,}\end{cases}$$

implies

$$\mathrm{ch}(\mathrm{Ind}_U^G(\psi_{(n)})) = \sum_{\lambda\in\mathcal{P}_n^\Theta}\tilde{\Psi}(s_\lambda)s_\lambda = \sum_{\substack{\lambda\in\mathcal{P}_n^\Theta\\ \mathrm{ht}(\lambda)=1}} s_\lambda. \qquad \square$$

### 5.3.3 Decomposition of $\mathrm{Ind}_U^G(\psi_\mu)$ (3)

Suppose $\lambda, \nu \in \mathcal{P}^\Theta$. A *column strict tableau $P$ of shape $\lambda$ and weight $\nu$* is a column strict filling of $\lambda$ such that for each $\varphi \in \Theta$,

$$\mathrm{sh}(P^{(\varphi)}) = \lambda^{(\varphi)} \qquad \text{and} \qquad \mathrm{wt}(P^{(\varphi)}) = \nu^{(\varphi)}.$$

We can now prove the theorem stated at the beginning of this section:

**Theorem 5.5 ([Zel81])** Let $U$ be the subgroup of unipotent upper-triangular matrices of $G = \mathrm{GL}_n(\mathbb{F}_q)$, $\mu \models n$ and $\psi_\mu$ be as in (4.8). Then

$$\mathrm{Ind}_U^G(\psi_\mu) = \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \mathrm{Card}(\hat{\mathcal{H}}_\mu^\lambda) G^\lambda.$$

*Proof.* Note that

$$\mathrm{Ind}_U^{P_\mu}(\psi_\mu) \cong \mathbb{C}P_\mu e_\mu = \mathbb{C}P_\mu e_{[\mu]} e'_{[\mu]},$$

where

$$e_{[\mu]} = \frac{1}{|U \cap L_\mu|} \sum_{u \in U \cap L_\mu} \psi_\mu(u^{-1}) u \qquad \text{and} \qquad e'_{[\mu]} = \frac{1}{|U_\mu|} \sum_{u \in U_\mu} u. \tag{5.13}$$

Thus,

$$\begin{aligned} \mathrm{Ind}_U^{P_\mu}(\psi_\mu) &\cong \mathrm{Inf}_{L_\mu}^{P_\mu}\left(\mathrm{Ind}_{U\cap L_\mu}^{L_\mu}(\psi_\mu)\right) \\ &\cong \mathrm{Inf}_{L_\mu}^{P_\mu}\left(\mathrm{Ind}_{U_{\mu_1}}^{G_{\mu_1}}(\psi_{(\mu_1)}) \otimes \mathrm{Ind}_{U_{\mu_2}}^{G_{\mu_2}}(\psi_{(\mu_2)}) \otimes \cdots \otimes \mathrm{Ind}_{U_{\mu_\ell}}^{G_{\mu_\ell}}(\psi_{(\mu_\ell)})\right) \end{aligned}$$

In particular, by the definition of multiplication in $R$ (5.8),

$$\Gamma_\mu = \mathrm{ch}(\mathrm{Ind}_U^G(\psi_\mu)) = \Gamma_{\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_\ell}, \quad \text{where} \quad \Gamma_{\mu_i} = \sum_{\lambda \in \mathcal{P}^\Theta_{\mu_i}, \mathrm{ht}(\lambda)=1} s_\lambda.$$

Pieri's rule (2.10) implies that for $\lambda \in \mathcal{P}^\Theta_r$, $\nu \in \mathcal{P}^\Theta_s$ and $\mathrm{ht}(\nu) = 1$,

$$s_\lambda s_\nu = \sum_{\gamma \in \mathcal{P}^\Theta_{r+s}, |\hat{\mathcal{H}}_\nu^{\gamma/\lambda}| \neq 0} s_\gamma, \qquad \text{so} \qquad \Gamma_\mu = \sum_{\lambda \in \mathcal{P}^\Theta} K_{\lambda\mu} s_\lambda,$$

where

$$\begin{aligned} K_{\lambda\mu} &= \mathrm{Card}\{\emptyset = \gamma_0 \subset \gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_\ell = \lambda \mid |\hat{\mathcal{H}}_{(\mu_{i+1})}^{\gamma_{i+1}/\gamma_i}| = 1\} \\ &= \mathrm{Card}\{\text{column strict tableaux of shape } \lambda \text{ and weight } \mu\} = |\hat{\mathcal{H}}_\mu^\lambda|. \end{aligned}$$

By Green's Theorem (Theorem 5.6), ch is an isomorphism, so

$$\mathrm{Ind}_U^G(\psi_\mu) = \mathrm{ch}^{-1}(\Gamma_\mu) = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} |\hat{\mathcal{H}}_\mu^\lambda| \mathrm{ch}^{-1}(s_\lambda) = \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \mathrm{Card}(\hat{\mathcal{H}}_\mu^\lambda) G^\lambda. \qquad \square$$

## 5.4 A weight space decomposition of $\mathcal{H}_\mu$-modules

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \models n$ and let $P_\mu$, $L_\mu$ and $U_\mu$ be as in (5.6) and (5.7). Recall that

$$e_\mu = \frac{1}{|U|}\sum_{u\in U}\psi_\mu(u^{-1})u.$$

**Theorem 5.8.** *For $a \in M_\mu$, let $T_a = e_\mu v_a e_\mu$ with $v_a$ as in (4.14). Then the map*

$$\begin{array}{ccc} \mathcal{H}_{(\mu_1)}\otimes\mathcal{H}_{(\mu_2)}\otimes\cdots\otimes\mathcal{H}_{(\mu_\ell)} & \longrightarrow & \mathcal{H}_\mu \\ T_{(f_1)}\otimes T_{(f_2)}\otimes\cdots\otimes T_{(f_\ell)} & \mapsto & T_{(f_1)\oplus(f_2)\oplus\cdots\oplus(f_\ell)}, \qquad \text{for } (f_i)\in M_{(\mu_i)} \end{array}$$

*is an injective algebra homomorphism with image $\mathcal{L}_\mu = e_\mu P_\mu e_\mu = e_\mu L_\mu e_\mu$.*

*Proof.* Note that

$$T_{(f_1)}\otimes\cdots\otimes T_{(f_\ell)} = \frac{1}{|U\cap L_\mu|^2}\sum_{x_i,y_i\in U_{\mu_i}}\left(\prod_{i=1}^{\ell}\psi_\mu(x_i^{-1}y_i^{-1})\right)x_1v_{(f_1)}y_1\otimes\cdots\otimes x_\ell v_{(f_\ell)}y_\ell.$$

Since $U = (L_\mu\cap U)(U_\mu)$, $L_\mu\cap U \cong U_{\mu_1}\times U_{\mu_2}\times\cdots\times U_{\mu_\ell}$, and $\psi_\mu$ is trivial on $U_\mu$,

$$\begin{aligned} T_{(f_1)\oplus(f_2)\oplus\cdots\oplus(f_\ell)} &= \frac{1}{|U|^2}\sum_{x,y\in U}\psi_\mu(x^{-1}y^{-1})x(v_{(f_1)}\oplus v_{(f_2)}\oplus\cdots\oplus v_{(f_\ell)})y \\ &= \frac{1}{|U\cap L_\mu|^2}\sum_{x_i,y_i\in U_{\mu_i}}\psi_\mu(x_1^{-1}y_1^{-1}\oplus\cdots\oplus x_\ell^{-1}y_\ell^{-1})e'_{[\mu]}x_1v_{(f_1)}y_1\oplus\cdots\oplus x_\ell v_{(f_\ell)}y_\ell e'_{[\mu]}, \end{aligned}$$

where $e'_{[\mu]}$ is as in (5.13). Since $L_\mu \subseteq N_G(U_\mu)$, the idempotent $e'_{[\mu]}$ commutes with $x_1v_{(f_1)}y_1\oplus\cdots\oplus x_\ell v_{(f_\ell)}y_\ell$ and

$$T_{(f_1)\oplus(f_2)\oplus\cdots\oplus(f_\ell)} = \frac{e'_{[\mu]}}{|L\cap U|^2}\sum_{x_i,y_i\in U_{\mu_i}}\left(\prod_{i=1}^{\ell}\psi_\mu(x_i^{-1}y_i^{-1})\right)x_1v_{(f_1)}y_1\oplus\cdots\oplus x_\ell v_{(f_\ell)}y_\ell.$$

Consequently, the map multiplies by $e'_{[\mu]}$ and changes $\otimes$ to $\oplus$, so it is an algebra homomorphism. Since the map sends basis elements to basis elements, it is also injective. □

**Remark.** This is the $GL_n(\mathbb{F}_q)$ version of Corollary 3.5.

Let $\mathcal{L}_\mu$ be as in Theorem 5.8. By Theorem 5.1 each $\mathcal{H}_{(\mu_i)}$ is commutative, so $\mathcal{L}_\mu$ is commutative and all the irreducible $\mathcal{L}_\mu$-modules $\mathcal{L}_\mu^\gamma$ are one-dimensional. Theorem 5.3 implies that

$$\hat{\mathcal{H}}_{(\mu_i)} = \{\Theta\text{-partitions } \lambda \mid |\lambda| = \mu_i, \mathrm{ht}(\lambda) = 1\}$$

indexes the irreducible $\mathcal{H}_{(\mu_i)}$-modules. Therefore, the set

$$\hat{\mathcal{L}}_\mu = \hat{\mathcal{H}}_{(\mu_1)}\times\hat{\mathcal{H}}_{(\mu_2)}\times\cdots\times\hat{\mathcal{H}}_{(\mu_\ell)} = \{\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_\ell) \mid \gamma_i\in\hat{\mathcal{H}}_{(\mu_i)}\} \tag{5.14}$$

indexes the irreducible $\mathcal{L}_\mu$-modules. Identify $\gamma \in \hat{\mathcal{L}}_\mu$ with the map $\gamma : \mathcal{L}_\mu \to \mathbb{C}$ such that

$$yv = \gamma(y)v, \qquad \text{for all } y \in \mathcal{L}_\mu, v \in \mathcal{L}_\mu^\gamma.$$

For $\gamma \in \hat{\mathcal{L}}_\mu$, the $\gamma$-*weight space* $V_\gamma$ of an $\mathcal{H}_\mu$-module $V$ is

$$V_\gamma = \{v \in V \mid yv = \gamma(y)v, \text{ for all } y \in \mathcal{L}_\mu\}.$$

Then

$$V \cong \bigoplus_{\gamma \in \hat{\mathcal{L}}_\mu} V_\gamma.$$

Let $\lambda \in \mathcal{P}^\Theta$ and $\gamma \in \hat{\mathcal{L}}_\mu$. A *column strict tableau $P$ of shape $\lambda$ and weight $\gamma$* is column strict filling of $\lambda$ such that for each $\varphi \in \Theta$,

$$\mathrm{sh}(P^{(\varphi)}) = \lambda^{(\varphi)} \qquad \text{and} \qquad \mathrm{wt}(P^{(\varphi)}) = (|\gamma_1^{(\varphi)}|, |\gamma_2^{(\varphi)}|, \ldots, |\gamma_\ell^{(\varphi)}|),$$

where $|\gamma_i^{(\varphi)}|$ is the number of boxes in the partition $\gamma_i^{(\varphi)}$ (which has length 1). Let

$$\hat{\mathcal{H}}_\gamma^\lambda = \{P \in \mathcal{H}_\mu^\lambda \mid \mathrm{sh}(P) = \lambda, \mathrm{wt}(P) = \gamma\}.$$

For example, suppose

$$\lambda = \left( \ydiagram{3,2}^{(\varphi_1)}, \ \ydiagram{2,1}^{(\varphi_2)}, \ \ydiagram{1,1}^{(\varphi_3)} \right)$$

and

$$\gamma = \left(\ydiagram{2}^{(\varphi_1)}, \ \ydiagram{1}^{(\varphi_2)}, \ \ydiagram{1}^{(\varphi_3)}\right) \otimes \left(\ydiagram{2}^{(\varphi_1)}, \ \ydiagram{1}^{(\varphi_3)}\right) \otimes \left(\ydiagram{1}^{(\varphi_1)}, \ \ydiagram{1}^{(\varphi_2)}\right) \otimes \left(\ydiagram{1}^{(\varphi_2)}\right).$$

Then

$$\hat{\mathcal{H}}_\gamma^\lambda = \left\{ \begin{array}{l} \left(\young(113,22)^{(\varphi_1)}, \ \young(13,4)^{(\varphi_2)}, \ \young(1,2)^{(\varphi_3)}\right), \left(\young(113,22)^{(\varphi_1)}, \ \young(14,3)^{(\varphi_2)}, \ \young(1,2)^{(\varphi_3)}\right) \\ \left(\young(112,23)^{(\varphi_1)}, \ \young(13,4)^{(\varphi_2)}, \ \young(1,2)^{(\varphi_3)}\right), \left(\young(112,23)^{(\varphi_1)}, \ \young(14,3)^{(\varphi_2)}, \ \young(1,2)^{(\varphi_3)}\right) \end{array} \right\}.$$

**Theorem 5.9.** *Let $\mathcal{H}_\mu^\lambda$ be an irreducible $\mathcal{H}_\mu$-module and $\gamma \in \hat{\mathcal{L}}_\mu$. Then*

$$\dim(\mathcal{H}_\mu^\lambda)_\gamma = \mathrm{Card}(\hat{\mathcal{H}}_\gamma^\lambda).$$

*Proof.* By Theorem 2.3 and Proposition 2.2,

$$\dim((\mathcal{H}_\mu^\lambda)_\gamma) = \langle \mathrm{Res}_{\mathcal{L}_\mu}^{\mathcal{H}_\mu}(\mathcal{H}_\mu^\lambda), \mathcal{L}_\mu^\gamma \rangle = \langle \mathrm{Res}_{P_\mu}^G(G^\lambda), P_\mu^\gamma \rangle = \langle G^\lambda, \mathrm{Ind}_{P_\mu}^G(P_\mu^\gamma) \rangle,$$

where $P_\mu^\gamma = \mathrm{Inf}_{L_\mu}^{P_\mu}(L_\mu^\gamma)$. Therefore,

$$\dim((\mathcal{H}_\mu^\lambda)_\gamma) = c_\gamma^\lambda, \qquad \text{where} \quad s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_\ell} = \sum_{\lambda \in \mathcal{P}^\Theta} c_\gamma^\lambda s_\lambda.$$

Pieri's rule (2.10) implies $c_\gamma^\lambda = |\hat{\mathcal{H}}_\gamma^\lambda|$. □

# Chapter 6

# The representation theory of the Yokonuma algebra

## 6.1 General type

Let $\mathbb{1} : U \to \mathbb{C}^*$ be the trivial character, and let

$$e_{\mathbf{1}} = \frac{1}{|U|} \sum_{u \in U} u \quad \in \mathbb{C}G \tag{6.1}$$

be the idempotent so that $\mathrm{Ind}_U^G(\mathbb{1}) \cong \mathbb{C}Ge_{\mathbf{1}}$. Then the Yokonuma algebra

$$\mathcal{H}_{\mathbf{1}} = \mathrm{End}_{\mathbb{C}G}(\mathbb{C}Ge_{\mathbf{1}}) \cong e_{\mathbf{1}}\mathbb{C}Ge_{\mathbf{1}}$$

has a basis

$$\{e_{\mathbf{1}}ve_{\mathbf{1}} \mid v \in N\}, \quad \text{indexed by} \quad N = \langle \xi_i, h \mid i = 1, 2, \dots, \ell,\ h \in T \rangle,$$

where $\xi_i = w_i(1) = x_i(1)x_{-\alpha_i}(-1)x_i(1)$, and $T = \langle h_H(t) \mid H \in \mathfrak{h}_Z \rangle$. Recall that $h_i(t) = h_{H_{\alpha_i}}(t)$.

The map

$$\begin{array}{ccc} \mathbb{C}T & \longrightarrow & \mathcal{H}_{\mathbf{1}} \\ h & \mapsto & e_{\mathbf{1}}he_{\mathbf{1}} \end{array}$$

is an injective algebra homomorphism (see Chapter 3 (E3)). For $v \in N$, write

$$T_v = e_{\mathbf{1}}ve_{\mathbf{1}} \in \mathcal{H}_{\mathbf{1}}, \qquad \text{with} \qquad T_i = T_{\xi_i}, \qquad \text{and} \qquad h = T_h, \quad h \in T.$$

**Theorem 6.1 (Yokonuma).** *$\mathcal{H}_{\mathbf{1}}$ is generated by $T_i$, $i = 1, \dots, \ell$, and $h \in T$ with relations*

$$\begin{aligned} T_i h &= s_i(h) T_i, \\ T_i^2 &= q^{-1} h_i(-1) + q^{-1} \sum_{t \in \mathbb{F}_q^*} h_i(t) T_i, \\ \underbrace{T_i T_j T_i \cdots}_{m_{ij}} &= \underbrace{T_j T_i T_j \cdots}_{m_{ij}}, \quad \text{where } m_{ij} \text{ is the order of } s_i s_j \text{ in } W, \\ T_h T_k &= T_{hk}, \quad h, k \in T. \end{aligned}$$

*Proof.* Consider the product $\xi_i\xi_j \in N$ for $i \neq j$. Note that

$$\begin{aligned} e_{\mathbf{1}}\xi_i\xi_j e_{\mathbf{1}} &= e_{\mathbf{1}}\Big(\prod_{\alpha\neq\alpha_i} e_\alpha\Big)\xi_i\xi_j e_{\alpha_i} e_{\mathbf{1}} && \text{(by (3.17))} \\ &= e_{\mathbf{1}}\xi_i\Big(\prod_{\alpha\neq\alpha_i} e_\alpha\Big)e_{\alpha_i} e\xi_j e_{\mathbf{1}} && \text{(by Chapter 3, (E1))} \\ &= e_{\mathbf{1}}\xi_i e_{\mathbf{1}}\xi_j e_{\mathbf{1}} = (e_{\mathbf{1}}\xi_i e_{\mathbf{1}})(e_{\mathbf{1}}\xi_j e_{\mathbf{1}}). \end{aligned}$$

Thus, if $v = v_1 v_2 \cdots v_r v_T \in N$ decomposes according to $s_{i_1}s_{i_2}\cdots s_{i_r} \in W$ (see (3.4)), then

$$T_v = e_{\mathbf{1}} v_1 v_2 \cdots v_r v_T e_{\mathbf{1}} = T_{i_1} T_{i_2} \cdots T_{i_r} v_T. \tag{6.2}$$

The Yokonuma algebra is therefore generated by $T_i$, $i = 1, 2, \ldots, \ell$, and $h \in T$.

The necessity of the relations is a direct consequence of (6.2), (N3), (UN3), and (N2) (the last three are from Chapter 3). For sufficiency, use a similar argument to the one used in the proof of Theorem 3.5 in [IM65]. □

### 6.1.1 A reduction theorem

Let $\hat{T}$ index the irreducible $\mathbb{C}T$-modules. Since $T$ is abelian, all the irreducible modules are one-dimensional, and we may identify the label $\gamma \in \hat{T}$ of $V^\gamma = \mathbb{C}\text{-span}\{v_\gamma\}$ with the homomorphism $\gamma : T \to \mathbb{C}^*$ given by

$$hv_\gamma = \gamma(h)v_\gamma, \qquad \text{for } h \in T.$$

Suppose $V$ is an $\mathcal{H}_{\mathbf{1}}$-module. As a $T$-module

$$V = \bigoplus_{\gamma\in\hat{T}} V_\gamma, \qquad \text{where } V_\gamma = \{v \in V \mid hv = \gamma(h)v,\ h \in T\}.$$

Note that

$$\mathcal{H}_{\mathbf{1}} = \bigoplus_{\gamma\in\hat{T}} \mathcal{H}_{\mathbf{1}}\tau_\gamma, \qquad \text{where} \quad \tau_\gamma = \frac{1}{|T|}\sum_{h\in T}\gamma(h^{-1})h. \tag{6.3}$$

Recall the surjection $\pi : N \to W$ of (3.3). Use this map to identify

$$w = s_{i_1}s_{i_2}\ldots s_{i_r} \in W \quad \longleftrightarrow \quad w = \xi_{i_1}\xi_{i_2}\ldots\xi_{i_r} \in N, \quad \text{for } r \text{ minimal}. \tag{6.4}$$

Thus, the $W$-action on $\hat{T}$

$$(w\gamma)(h) = \gamma(w^{-1}(h)), \qquad \text{for } w \in W,\ h \in T, \text{ and } \gamma \in \hat{T},$$

implies that for $\gamma \in \hat{T}$, $v_\gamma \in V_\gamma$,

$$hT_w v_\gamma = T_w w^{-1}(h) v_\gamma = \gamma(w^{-1}h) T_w v_\gamma \qquad \text{for all } h \in T.$$

Thus,

$$T_w(V_\gamma) \subseteq V_{w\gamma}. \tag{6.5}$$

Let

$$W_\gamma = \{w \in W \mid w(\gamma) = \gamma\}$$

and

$$\mathcal{T}_\gamma = \tau_\gamma \mathcal{H}_{\mathbf{1}} \tau_\gamma = \mathbb{C}\text{-span}\{\tau_\gamma T_w \tau_\gamma \mid w \in W_\gamma\}. \tag{6.6}$$

**Remarks**

1. Since the identity of $\mathcal{H}_{\mathbf{1}}$ is $e_{\mathbf{1}}$ and the identity of $\mathcal{T}_\gamma$ is $\tau_\gamma$, the algebra $\mathcal{T}_\gamma$ is not a subalgebra of $\mathcal{H}_{\mathbf{1}}$. In fact, $\mathcal{T}_\gamma \cong \mathrm{End}_{\mathcal{H}_1}(\mathrm{Ind}_{\mathbb{C}T}^{\mathcal{H}_1}(\gamma))$.
2. If $V$ is an $\mathcal{H}_{\mathbf{1}}$-module, then

$$\tau_\gamma V = V_\gamma, \quad \text{and so} \quad \mathcal{T}_\gamma V = \mathcal{T}_\gamma V_\gamma.$$

   In particular, $V_\gamma$ is an $\mathcal{T}_\gamma$-module.

**Theorem 6.2.** *Let $V$ be an irreducible $\mathcal{H}_{\mathbf{1}}$-module such that $V_\gamma \neq 0$. Then*

*(a) $V_\gamma$ is an irreducible $\mathcal{T}_\gamma$-module;*

*(b) if $M$ is an irreducible $\mathcal{T}_\gamma$-module, then $\mathcal{H}_{\mathbf{1}} \otimes_{\mathcal{T}_\gamma} M$ is an irreducible $\mathcal{H}_{\mathbf{1}}$-module;*

*(c) the map*

$$\begin{array}{rcl} \mathrm{Ind}_{\mathcal{T}_\gamma}^{\mathcal{H}_1}(V_\gamma) = \mathcal{H}_{\mathbf{1}} \otimes_{\mathcal{T}_\gamma} V_\gamma & \xrightarrow{\sim} & V \\ T_w \otimes v_\gamma & \mapsto & T_w v_\gamma \end{array}$$

*is an $\mathcal{H}_{\mathbf{1}}$-module isomorphism.*

*Proof.* (a) Let $0 \neq v_\gamma \in V_\gamma$. The irreducibility of $V$ implies that $\mathcal{H}_{\mathbf{1}} v_\gamma = V$, so for any $v \in V_\gamma$, there exist elements $c_{w,\nu} \in \mathbb{C}$ such that

$$\begin{aligned} v &= \sum_{\substack{w \in W \\ \nu \in \hat{T}}} c_{w,\nu} T_w \tau_\nu v_\gamma && \text{(by (6.3))} \\ &= \sum_{w \in W} c_{w,\gamma} T_w \tau_\gamma v_\gamma && \text{(by the orthogonality of characters of } T) \\ &= \sum_{w \in W_\gamma} c_{w,\gamma} \tau_\gamma T_w \tau_\gamma v_\gamma. && \text{(by (6.5) and since } v \in V_\gamma) \end{aligned}$$

Since an arbitrarily chosen $v$ is contained in $\mathcal{T}_\gamma v_\gamma$, we have $\mathcal{T}_\gamma v_\gamma = V_\gamma$, making $V_\gamma$ an irreducible $\mathcal{T}_\gamma$-module.

(b) Suppose $M$ is an irreducible $\mathcal{T}_\gamma$-module. Let

$$\begin{aligned} V = \mathrm{Ind}_{\mathcal{T}_\gamma}^{\mathcal{H}_1}(M) &= \mathcal{H}_{\mathbf{1}} \otimes_{\mathcal{T}_\gamma} M \\ &= \mathbb{C}\text{-span}\{T_w \tau_\nu \otimes v \mid w \in W, \nu \neq \gamma \in \hat{T}, v \in M\} \\ &= \mathbb{C}\text{-span}\{T_w \otimes v \mid w \in W/W_\gamma, v \in M\} \end{aligned}$$

where the last equality follows from

$$\tau_\nu \otimes v = \tau_\nu \otimes \tau_\gamma v = \tau_\nu \tau_\gamma \otimes v = 0 \otimes v, \quad \text{for } \nu \neq \gamma.$$

Note that $V_\gamma = \mathbb{C}\text{-span}\{e_{\mathbf{1}} \otimes v \mid v \in M\} \cong M$. In particular, $V_\gamma$ is an irreducible $\mathcal{T}_\gamma$-module and $\mathcal{H}_{\mathbf{1}} V_\gamma = V$.

Suppose $V$ has a nontrivial $\mathcal{H}_{\mathbf{1}}$-submodule $V'$. Since $V$ is induced from $V_\gamma$, there must by some $T_w \in \mathcal{H}_{\mathbf{1}}$ such that

$$T_w(V') \cap V_\gamma \neq 0.$$

But $V'$ is an $\mathcal{H}_{\mathbf{1}}$-module, so $V' \cap V_\gamma \neq 0$. As an irreducible $\mathcal{T}_\gamma$-module $V_\gamma \subseteq V'$, and by the construction of $V$,

$$\mathcal{H}_{\mathbf{1}} V' \supseteq \mathcal{H}_{\mathbf{1}} V_\gamma = V.$$

Therefore, $V$ contains no nontrivial, *proper* submodules, making $V$ an irreducible $\mathcal{H}_{\mathbf{1}}$-module.

(c) Follows from the proof of (b). □

Let $\hat{T}/W$ be the set of $W$-orbits in $\hat{T}$. Identify $\hat{T}/W$ with a set of orbit representatives, and for $\gamma \in \hat{T}/W$, let $\hat{\mathcal{T}}_\gamma$ index the irreducible modules of $\mathcal{T}_\gamma$.

**Corollary 6.3.** *The map*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \textit{Irreducible} \\ \mathcal{H}_{\mathbf{1}}\textit{-modules} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \textit{Pairs } (\gamma, \lambda), \\ \gamma \in \hat{T}/W, \lambda \in \hat{\mathcal{T}}_\gamma \end{array} \right\} \\ \mathrm{Ind}_{\mathcal{T}_\gamma}^{\mathcal{H}_1}(\mathcal{T}_\gamma^\lambda) & \leftrightarrow & (\gamma, \lambda) \end{array}$$

*is a bijection.*

### 6.1.2 The algebras $\mathcal{T}_\gamma$

Let $B = UT$ be a Borel subgroup of $G$. Since

$$\begin{array}{ccc} B & \longrightarrow & T \\ uh & \mapsto & h \end{array}$$

is a surjective homomorphism, $\gamma : T \to \mathbb{C}^*$ extends to a linear character $\gamma$ of $B$ given by $\gamma(uh) = \gamma(h)$. Note that

$$e_{\mathbf{1}}\tau_\gamma = \frac{1}{|B|}\sum_{b\in B}\gamma(b^{-1})b \quad (e_{\mathbf{1}} \text{ as in (6.1)}).$$

Thus,

$$\mathcal{T}_\gamma = \tau_\gamma e_{\mathbf{1}}\mathbb{C}Ge_{\mathbf{1}}\tau_\gamma \cong \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G(\gamma)) = \mathcal{H}(G, B, \gamma),$$

where if $\gamma$ is the trivial character $1_B$, then $\mathcal{H}(G, B, 1_B)$ is the Iwahori-Hecke algebra.

**Proposition 6.4.** *Let $\gamma \in \hat{T}$ be such that $W_\gamma$ is generated by simple reflections. Then $\mathcal{T}_\gamma$ is presented by generators $\{T_i \mid s_i \in W_\gamma\}$ with relations*

$$T_i^2 = \begin{cases} q^{-1} + q^{-1}(q-1)T_i, & \text{if } \gamma(h_{\alpha_i}(t)) = 1 \text{ for all } t \in \mathbb{F}_q^*, \\ \gamma(h_{\alpha_i}(-1))q^{-1}, & \text{otherwise,} \end{cases}$$

$$\underbrace{T_iT_jT_i\cdots}_{m_{ij}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}}, \quad \text{where } m_{ij} \text{ is the order of } s_is_j \text{ in } W.$$

*Proof.* This follows from Theorem 6.1 and

$$\Big(\sum_{t\in\mathbb{F}_q^*} h_{\alpha_i}(t)\Big)\tau_\gamma = \sum_{t\in\mathbb{F}_q^*}\gamma(h_{\alpha_i}(t))\tau_\gamma = \begin{cases} (q-1)\tau_\gamma, & \text{if } \gamma(h_{\alpha_i}(t)) = 1 \text{ for all } t \in \mathbb{F}_q^*, \\ 0, & \text{otherwise.} \end{cases}$$

□

## 6.2 The $G = GL_n(\mathbb{F}_q)$ case

### 6.2.1 The Yokonuma algebra and the Iwahori-Hecke algebra

If $\psi_\mu = \mathbb{1}$ is the trivial character of $U$, then $\mu = (1^n)$. Let $T_i = e_{\mathbf{1}}s_ie_{\mathbf{1}}$ and recall that $h_{\varepsilon_j}(t) = Id_{j-1} \oplus (t) \oplus Id_{n-j}$. In the case $G = GL_n(\mathbb{F}_q)$, $\mathcal{H}_{\mathbf{1}}$ has generators $T_i$, $h_{\varepsilon_j}(t)$, for $1 \le i < n$, $1 \le j \le n$ and $t \in \mathbb{F}_q^*$ with relations

$$T_i^2 = q^{-1} + q^{-1}h_{\varepsilon_i}(-1)\sum_{t\in\mathbb{F}_q^*} h_{\varepsilon_i}(t)h_{\varepsilon_{i+1}}(t^{-1})T_i,$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_iT_j = T_jT_i, \quad |i-j| > 1,$$

$$h_{\varepsilon_j}(t)T_i = T_ih_{\varepsilon_{j'}}(t), \qquad \text{where } j' = s_i(j),$$

$$h_{\varepsilon_i}(a)h_{\varepsilon_i}(b) = h_{\varepsilon_i}(ab), \qquad h_{\varepsilon_i}(a)h_{\varepsilon_j}(b) = h_{\varepsilon_j}(b)h_{\varepsilon_i}(a), \quad a, b \in \mathbb{F}_q^*.$$

The Yokonuma algebra has a decomposition

$$\mathcal{H}_{\mathbf{1}} = \bigoplus_{\gamma\in\hat{T}}\mathcal{H}_{\mathbf{1}}\tau_\gamma, \qquad \text{where} \quad \tau_\gamma = \frac{1}{|T|}\sum_{h\in T}\gamma(h^{-1})h.$$

Fix a total ordering $\le$ on the set $\hat{\mathbb{F}}_q^* = \{\varphi_1, \varphi_2, \ldots, \varphi_{q-1}\}$ of linear characters of $\mathbb{F}_q^*$, such that $\varphi_1 : \mathbb{F}_q^* \to \mathbb{C}^*$ is the trivial character. Suppose $\gamma \in \hat{T}$. Since $\langle h_{\varepsilon_i}(t) \mid t \in \mathbb{F}_q^* \rangle \cong \mathbb{F}_q^*$,

$$\begin{aligned} \gamma(h) &= \gamma(h_{\varepsilon_1}(h_1))\gamma(h_{\varepsilon_2}(h_2))\cdots\gamma(h_{\varepsilon_n}(h_n)), &\quad& \text{where } h = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in T, \\ &= \gamma_1(h_1)\gamma_2(h_2)\cdots\gamma_n(h_n), && \text{where } \gamma_i \in \hat{\mathbb{F}}_q^*. \end{aligned}$$

Thus, every $\gamma \in \hat{T}$ can be written $\gamma = \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_n}$.

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \models n$ with $\nu_i \ge 0$. Write

$$\gamma_\nu = \underbrace{\varphi_1 \otimes \cdots \otimes \varphi_1}_{\nu_1 \text{ terms}} \otimes \underbrace{\varphi_2 \otimes \cdots \otimes \varphi_2}_{\nu_2 \text{ terms}} \otimes \cdots \otimes \underbrace{\varphi_n \otimes \cdots \otimes \varphi_n}_{\nu_n \text{ terms}} \quad \text{and} \quad \tau_\nu = \tau_{\gamma_\nu}. \tag{6.7}$$

Note that the $\gamma_\nu$ are orbit representatives of the $W$-action in $\hat{T}$. Let

$$\begin{aligned} W_\nu &= \{w \in W \mid w(\gamma_\nu) = \gamma_\nu\} = W_{\nu_1} \oplus W_{\nu_2} \oplus \cdots \oplus W_{\nu_n}, \\ \mathcal{T}_\nu &= \tau_\nu \mathcal{H}_{\mathbf{1}} \tau_\nu = \mathbb{C}\text{-span}\{\tau_\nu T_w \tau_\nu \mid w \in W_\nu\}. \end{aligned}$$

Note that $W_\nu$ is generated by its simple reflections.

If $s_i$ is in the $k$th factor of $W_\nu = W_{\nu_1} \oplus W_{\nu_2} \oplus \cdots \oplus W_{\nu_n}$, then write

$$s_i \in W_{\nu_k}.$$

**Lemma 6.5.** *Let $\nu \models n$ and ${}^\nu T_w = \tau_\nu T_w \tau_\nu$ for $w \in W_\nu$. Then $\mathcal{T}_\nu$ is presented by generators*

$$\{{}^\nu T_i \mid s_i \in W_\nu\}$$

*with relations*

$$\begin{aligned} {}^\nu T_i^2 &= q^{-1} + \varphi_k(-1) q^{-1}(q-1)\,{}^\nu T_i, \quad \textit{for } s_i \in W_{\nu_k} \\ {}^\nu T_i\,{}^\nu T_{i+1}\,{}^\nu T_i &= {}^\nu T_{i+1}\,{}^\nu T_i\,{}^\nu T_{i+1} \\ {}^\nu T_i\,{}^\nu T_j &= {}^\nu T_j\,{}^\nu T_i, \qquad \textit{for } |i-j| > 1. \end{aligned}$$

*Proof.* Note that if $s_i \in W_\nu$ then $\gamma_\nu(h_{\varepsilon_i}(t)) = \gamma_\nu(h_{\varepsilon_{i+1}}(t))$. Thus, the lemma follows from the Yokonuma algebra relations. □

The *Iwahori-Hecke algebra* $\mathcal{T} \subseteq \mathcal{H}_{\mathbf{1}}$ is the algebra

$$\mathcal{T}_n = \mathcal{H}_{(n)}, \qquad \text{(recall } \varphi_1 \text{ is the trivial character).}$$

In $\mathcal{T}_n$, write

$$I_i = \tau_{(n)} T_i \tau_{(n)}.$$

**Corollary 6.6.** *Let $\nu = (\nu_1, \ldots, \nu_n) \models n$. Index the generators $I_i$ of $\mathcal{T}_{\nu_k}$ by $\{i \mid s_i \in W_{\nu_k}\}$. Then the map*

$$\begin{array}{ccc} \mathcal{T}_\nu & \xrightarrow{\sim} & \mathcal{T}_{\nu_1} \otimes \mathcal{T}_{\nu_2} \otimes \cdots \otimes \mathcal{T}_{\nu_n} \\ {}^\nu T_i & \mapsto & \varphi_k(-1) \underbrace{1 \otimes \cdots \otimes 1}_{k-1 \text{ terms}} \otimes I_i \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-k \text{ terms}}, \quad \text{for } s_i \in W_{\nu_k}, \end{array}$$

*is an algebra isomorphism.*

*Proof.* Let

$$\nu^{(k)} = (\underbrace{0, \ldots, 0}_{k-1 \text{ terms}}, \nu_k, \underbrace{0, \ldots, 0}_{n-k \text{ terms}}) \models \nu_k.$$

Then the map

$$\begin{array}{ccc} \mathcal{T}_{\nu^{(k)}} & \longrightarrow & \mathcal{T}_{\nu_k} \\ {}^{\nu^{(k)}} T_i & \mapsto & \varphi_k(-1) I_i \end{array}$$

is an algebra isomorphism by Lemma 6.5. Corollary 6.6 follows by applying this map to tensor products. □

## 6.2.2 The representation theory of $\mathcal{T}_\nu$

By Corollary 6.6, understanding the representation theory of $\mathcal{T}_\nu$ is the same as understanding the representation theory of the Iwahori-Hecke algebras $\mathcal{T}_{\nu_i}$.

Let $\mu \vdash n$ be a partition. A *standard tableau of shape $\mu$* is a column strict tableau of shape $\mu$ and weight $(1^n)$ (i.e. every number between from 1 to $n$ appears exactly once). Suppose $P$ is a standard tableau of shape $\mu$. Let

$$\begin{aligned} c_P(i) &= \text{ the content of the box containing } i, \\ C_P(i) &= \frac{q-1}{1 - q^{c_P(i) - c_P(i+1)}}. \end{aligned} \tag{6.8}$$

**Theorem 6.7 ([Hoe74, Ram97, Wen88]).** *Let $G = GL_n(\mathbb{F}_q)$. Then*

(a) *The irreducible $\mathcal{T}_n$-modules $\mathcal{T}_n^\mu$ are indexed by partitions $\mu \vdash n$,*

(b) $\dim(\mathcal{T}_n^\mu) = \text{Card}\{$*standard tableau* $P \mid \text{sh}(P) = \mu\}$,

(c) *Let $\mathcal{T}_n^\mu = \mathbb{C}\text{-span}\{v_P \mid \text{sh}(P) = \mu, \text{wt}(P) = (1^n)\}$. Then*

$$I_i v_P = q^{-1} C_P(i) v_P + q^{-1}(1 + C_P(i)) v_{s_i P}$$

*where $v_{s_i P} = 0$ if $s_i P$ is not a column strict tableau.*

Recall from Chapter 5, an $\hat{\mathbb{F}}_q^*$-partition $\lambda = (\lambda^{(\varphi_1)}, \lambda^{(\varphi_2)}, \ldots, \lambda^{(\varphi_{q-1})})$ is a sequence of partitions indexed by $\hat{\mathbb{F}}_q^*$. Let

$$|\lambda| = |\lambda^{(\varphi_1)}| + |\lambda^{(\varphi_2)}| + \cdots + |\lambda^{(\varphi_{q-1})}| = \text{ total number of boxes,}$$

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \models n$ with $\nu_i \geq 0$ and $\gamma_\nu$ as in (6.7). A *standard $\nu$-tableau*

$$Q = (Q^{(\varphi_1)}, Q^{(\varphi_2)}, \ldots, Q^{(\varphi_{q-1})})$$

*of shape* $\lambda$ is a column strict filling of $\lambda$ by the numbers $1, 2, 3, \ldots, n$ such that

(a) each number appears exactly once,

(b) $j \in Q^{(\varphi_k)}$ if $s_j \in W_{\nu_k}$.

Let

$$\begin{aligned} \hat{\mathcal{T}}_\nu^\lambda &= \{\text{standard } \nu\text{-tableau } Q \mid \text{sh}(Q) = \lambda\}, &(6.9)\\ \hat{\mathcal{T}}_\nu &= \{\hat{\mathbb{F}}_q^*\text{-partition } \lambda \mid \hat{\mathcal{T}}_\nu^\lambda \neq \emptyset\} \\ &= \{\hat{\mathbb{F}}_q^*\text{-partition } \lambda \mid |\lambda^{(\varphi_i)}| = |\nu_i|\}. &(6.10)\end{aligned}$$

**Example.** If $\nu = (3, 0, 1, 0)$, then $\gamma_\nu = \varphi_1 \otimes \varphi_1 \otimes \varphi_1 \otimes \varphi_3$. The set $\hat{\mathcal{T}}_\nu$ is

$$\left\{ \left(\ydiagram{3}^{(\varphi_1)}, \emptyset^{(\varphi_2)}, \ydiagram{1}^{(\varphi_3)}, \emptyset^{(\varphi_4)}\right), \left(\ydiagram{2,1}^{(\varphi_1)}, \emptyset^{(\varphi_2)}, \ydiagram{1}^{(\varphi_3)}, \emptyset^{(\varphi_4)}\right), \left(\ydiagram{1,1,1}^{(\varphi_1)}, \emptyset^{(\varphi_2)}, \ydiagram{1}^{(\varphi_3)}, \emptyset^{(\varphi_4)}\right) \right\},$$

and, for example,

$$\hat{\mathcal{T}}_\nu^{\left(\ydiagram{2,1}^{(\varphi_1)}, \ydiagram{1}^{(\varphi_3)}\right)} = \left\{ \left(\ytableaushort{12,3}^{(\varphi_1)}, \emptyset^{(\varphi_2)}, \ytableaushort{4}^{(\varphi_3)}, \emptyset^{(\varphi_4)}\right), \left(\ytableaushort{13,2}^{(\varphi_1)}, \emptyset^{(\varphi_2)}, \ytableaushort{4}^{(\varphi_3)}, \emptyset^{(\varphi_4)}\right) \right\}.$$

Let $Q \in \hat{\mathcal{T}}_\nu^\lambda$. Suppose $s_j \in W_{\nu_k}$ so that $j \in Q^{(\varphi_k)}$. Write

$$\begin{aligned} Q_j &= \varphi_k, &(6.11)\\ c_Q(j) &= \text{ the content of the box containing } j \text{ in } Q^{(\varphi_k)}, &(6.12)\\ C_Q(j) &= \frac{q-1}{1 - q^{c_Q(j) - c_Q(j+1)}} &(6.13)\end{aligned}$$

**Corollary 6.8.** *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \models n$ with $\nu_i \geq 0$ and $\ell(\nu) = n$. Then*

*(a) The irreducible $\mathcal{T}_\nu$-modules $\mathcal{T}_\nu^\lambda$ are indexed by $\lambda \in \hat{\mathcal{T}}_\nu$.*

*(b)* $\dim(\mathcal{T}_\nu^\lambda) = \text{Card}(\hat{\mathcal{T}}_\nu^\lambda)$.

*(c) Let $\mathcal{T}_\nu^\lambda = \mathbb{C}\text{-span}\{v_Q \mid Q \in \hat{\mathcal{T}}_\nu^\lambda\}$. Then*

$${}^\nu T_i v_Q = q^{-1} Q_j(-1) C_Q(i) v_Q + q^{-1} Q_j(-1)(1 + C_Q(i)) v_{s_i Q},$$

*where $v_{s_i Q} = 0$ if $s_i Q$ is not a column strict tableau.*

*Proof.* Transfer the explicit action of Theorem 6.7 across the isomorphism $\mathcal{T}_\nu \cong \mathcal{T}_{\nu_1} \otimes \mathcal{T}_{\nu_2} \otimes \cdots \otimes \mathcal{T}_{\nu_n}$ of Corollary 6.6. $\square$

### 6.2.3 The irreducible modules of $\mathcal{H}_{\mathbb{1}}$

The goal of this section is to construct the irreducible $\mathcal{H}_{\mathbf{1}}$-modules. According to Corollary 6.3 and (6.7), the map

$$\begin{matrix} \left\{ \begin{matrix} \text{Pairs } (\nu, \lambda), \nu \models n, \\ \nu_i \geq 0, \ell(\nu) = n, \lambda \in \hat{\mathcal{T}}_\nu \end{matrix} \right\} & \longleftrightarrow & \left\{ \begin{matrix} \text{Irreducible} \\ \mathcal{H}_{\mathbf{1}}\text{-modules} \end{matrix} \right\} \\ (\nu, \lambda) & \leftrightarrow & \mathrm{Ind}_{\mathcal{T}_\nu}^{\mathcal{H}_1}(\mathcal{T}_\nu^\lambda) \end{matrix} \tag{6.14}$$

is a bijection.

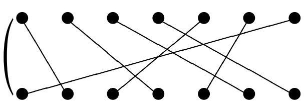
Let $W/\nu$ be the set of minimal length coset representatives of $W/W_\nu$. Then

$$\begin{aligned} \mathrm{Ind}_{\mathcal{T}_\nu}^{\mathcal{H}_1}(\mathcal{T}_\nu^\lambda) &= \mathcal{H}_{\mathbf{1}} \otimes_{\mathcal{T}_\nu} \mathcal{T}_\nu^\lambda \\ &= \mathbb{C}\text{-span}\{T_w \otimes v_Q \mid w \in W/\nu, \mathrm{sh}(Q) = \lambda, \mathrm{wt}(Q) = \gamma_\nu\}. \end{aligned} \tag{6.15}$$

Note that the map

$$\begin{matrix} \left\{ \begin{matrix} \text{Pairs } (w, Q), \\ w \in W/\nu, \ Q \in \hat{T}_\nu^\lambda \end{matrix} \right\} & \longleftrightarrow & \{\text{tableau } Q \mid \mathrm{sh}(Q) = \lambda, \mathrm{wt}(Q) = (1^n)\} \\ (w, Q) & \mapsto & wQ \end{matrix} \tag{6.16}$$

is a bijection (since $w \in W/\nu$ implies $w$ preserves the relative magnitudes of the entries in $Q^{(\varphi)}$ for all $\varphi$). For example, under this bijection



$$\left( \;\cdot\;, \left( \boxed{\begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}}^{(\varphi_1)}, \boxed{\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}}^{(\varphi_2)}, \boxed{7}^{(\varphi_3)} \right) \right) \quad \leftrightarrow \quad \left( \boxed{\begin{smallmatrix} 2 & 6 \\ 4 & 7 \end{smallmatrix}}^{(\varphi_1)}, \boxed{\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}}^{(\varphi_2)}, \boxed{1}^{(\varphi_3)} \right)$$

(since $2 < 3$ in $Q^{(\varphi_1)}$ on the left side, $4 < 6$ in $Q^{(\varphi_1)}$ on the right side). Write

$$v_{wQ} = T_w \otimes v_Q.$$

If $|\lambda| = n$, then let

$$\hat{\mathcal{H}}_{\mathbf{1}}^\lambda = \{ \text{ tableau } Q \mid \mathrm{sh}(Q) = \lambda, \mathrm{wt}(Q) = (1^n)\}, \tag{6.17}$$

$$\hat{\mathcal{H}}_{\mathbf{1}} = \{\hat{\mathbb{F}}_q^*\text{-partitions } \lambda \mid |\lambda| = n\}. \tag{6.18}$$

**Remarks.**

1. In the notation of Chapter 5,

$$\hat{\mathcal{H}}_{\mathbf{1}} = \{\Theta\text{-partition } \lambda \mid \hat{\mathcal{H}}_{\mathbf{1}}^\lambda \neq \emptyset\}.$$

2. The map

$$\begin{matrix} \left\{ \begin{matrix} \text{Pairs } (\nu, \lambda), \nu \models n, \\ \nu_i \geq 0, \ell(\nu) = n, \lambda \in \hat{\mathcal{T}}_\nu \end{matrix} \right\} & \longleftrightarrow & \hat{\mathcal{H}}_{\mathbf{1}} \\ (\nu, \lambda) & \leftrightarrow & \lambda \end{matrix}$$

is a bijection, so by (6.14), $\hat{\mathcal{H}}_{\mathbf{1}}$ indexes the irreducible $\mathcal{H}_{\mathbf{1}}$-modules.

Suppose $Q$ is a tableau of shape $\lambda$ and weight $(1^n)$. As in (6.11)-( 6.13), if $Q^{(\varphi)}$ has the box containing $j$, then write

$$\begin{aligned} Q_j &= \varphi, \\ c_Q(j) &= \text{ the content of the box containing } j \text{ in } Q^{(\varphi)}, \\ C_Q(j) &= \frac{q-1}{1-q^{c_Q(j)-c_Q(j+1)}}. \end{aligned}$$

**Theorem 6.9.**

*(a) The irreducible $\mathcal{H}_{\mathbf{1}}$-modules $\mathcal{H}_{\mathbf{1}}^{\lambda}$ are indexed by $\lambda \in \hat{\mathcal{H}}_{\mathbf{1}}$.*

*(b)* $\dim(\mathcal{H}_{\mathbf{1}}^{\lambda}) = \mathrm{Card}(\hat{\mathcal{H}}_{\mathbf{1}}^{\lambda})$.

*(c) Let $\mathcal{H}_{\mathbf{1}}^{\lambda} = \mathbb{C}\text{-span}\{v_Q \mid Q \in \hat{\mathcal{H}}_{\mathbf{1}}^{\lambda}\}$ as in (6.15) and (6.17). Then*

$$hv_Q = Q_1(h_1)Q_2(h_2)\cdots Q_n(h_n)v_Q, \qquad \textit{for } h = \mathrm{diag}(h_1, h_2, \ldots, h_n) \in T,$$

$$T_i v_Q = \begin{cases} q^{-1}v_{s_iQ}, & \textit{if } Q_i \succ Q_{i+1}, \\ Q_i(-1)q^{-1}C_Q(i)v_Q + Q_i(-1)q^{-1}(1+C_Q(i))v_{s_iQ}, & \textit{if } Q_i = Q_{i+1}, \\ v_{s_iQ}, & \textit{if } Q_i \prec Q_{i+1}, \end{cases}$$

*where $v_{s_iQ} = 0$ if $s_iQ$ is not a column strict tableau.*

*Proof.* (a) follows from Remark 2, and (b) follows from (6.15) and (6.17).
(c) Directly compute the action of $\mathcal{H}_{\mathbf{1}}$ on

$$\mathcal{H}_{\mathbf{1}}^{\lambda} = \mathrm{Ind}_{\mathcal{T}_\nu}^{\mathcal{H}_1}(\mathcal{T}_\nu^{\lambda}) = \mathbb{C}\text{-span}\{T_w \otimes v_Q \mid w \in W/\tilde{\lambda}, Q \in \hat{\mathcal{T}}_\nu^{\lambda}\}$$

For $1 \le i < n$, if $\ell(s_iw) < \ell(w)$, then

$$\begin{aligned} T_iT_w \otimes v_Q &= T_i^2 T_{s_iw} \otimes v_Q \\ &= q^{-1}T_{s_iw} \otimes v_Q + q^{-1}h_{\varepsilon_i}(-1)\sum_{t\in\mathbb{F}_q^*} h_{\varepsilon_i}(t)h_{\varepsilon_{i+1}}(t^{-1})T_w \otimes v_Q \\ &= q^{-1}T_{s_iw} \otimes v_Q + q^{-1}(wQ)_i(-1)\sum_{t\in\mathbb{F}_q^*}(wQ)_i(t)(wQ)_{i+1}(t^{-1})T_w \otimes v_Q \end{aligned}$$

Since $\ell(s_iw) < \ell(w)$ can hold only if $(wQ)_i \neq (wQ)_{i+1}$, the orthogonality of characters implies

$$T_iT_w \otimes v_Q = q^{-1}T_{s_iw} \otimes v_Q + 0. \tag{6.19}$$

If $\ell(s_iw) > \ell(w)$, then there are two cases:
Case 1: $\ell(s_iws_j) < \ell(s_iw)$ for some $s_j \in W_{\tilde{\gamma}}$,
Case 2: $\ell(s_iws_j) > \ell(s_iw)$ for all $s_j \in W_{\tilde{\gamma}}$.

In Case 1,

$$\begin{aligned} T_iT_w \otimes v_Q &= T_{s_iw} \otimes v_Q = T_{s_iws_j}T_j \otimes v_Q = T_{s_iws_j} \otimes T_jv_Q \\ &= T_{s_iws_j} \otimes Q_j(-1)q^{-1}C_Q(j)v_Q + Q_j(-1)q^{-1}(1 + C_Q(j))v_{s_jQ} \\ &= Q_j(-1)q^{-1}C_Q(j)T_{s_iws_j} \otimes v_Q + Q_j(-1)q^{-1}(1 + C_Q(j))T_{s_iws_j} \otimes v_{s_jQ}. \end{aligned} \tag{6.20}$$

In Case 2,

$$T_iT_w \otimes v_Q = T_{s_iw} \otimes v_Q. \tag{6.21}$$

Make the identification of (6.16) in equations (6.19), (6.20), and (6.21) to obtain the $\mathcal{H}_{\mathbf{1}}$-action on $\mathcal{H}_{\mathbf{1}}^{\lambda}$. □

**Sample Computations**. If $n = 10$ for $\mathcal{H}_{\mathbf{1}}$, then

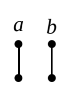
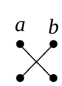
$$T_1 v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)} = v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 2&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 1\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)},$$

$$T_2 v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)} = q^{-1} v_{\left(\begin{array}{|c|c|c|}\hline 2&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 3\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)},$$

$$T_3 v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)} = -\varphi_1(-1)q^{-1} v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)} + 0,$$

$$\begin{aligned} T_4 v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)} &= \frac{\varphi_1(-1)q}{q+1} v_{\left(\begin{array}{|c|c|c|}\hline 3&5&10\\ \hline 4&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)} \\ &\quad + \varphi_1(-1)\left(q^{-1} + \frac{q}{q+1}\right) v_{\left(\begin{array}{|c|c|c|}\hline 3&4&10\\ \hline 5&7\\ \cline{1-2}\end{array}^{(\varphi_1)}, \begin{array}{|c|c|}\hline 1&6\\ \hline 8\\ \cline{1-1}\end{array}^{(\varphi_2)}, \begin{array}{|c|}\hline 2\\ \hline 9\\ \hline\end{array}^{(\varphi_3)}\right)}. \end{aligned}$$

A Character Table $\mathcal{H}_{\mathbf{1}}$ for $n = 2$:

| $\mathcal{H}_{\mathbf{1}}$ |  |  |
|---|---|---|
| $\left(\begin{array}{|c|c|}\hline\ &\ \\ \hline\end{array}^{(\varphi_i)}\right)$ | $\varphi_i(ab)$ | $\varphi_i(-ab)$ |
| $\left(\begin{array}{|c|}\hline\ \\ \hline\ \\ \hline\end{array}^{(\varphi_i)}\right)$ | $\varphi_i(ab)$ | $-\varphi_i(-ab)q^{-1}$ |
| $\left(\begin{array}{|c|}\hline\ \\ \hline\end{array}^{(\varphi_i)}, \begin{array}{|c|}\hline\ \\ \hline\end{array}^{(\varphi_j)}\right)$ | $\varphi_i(a)\varphi_j(b) + \varphi_i(b)\varphi_j(a)$ | 0 |

# Appendix A

# Commutation Relations

The following relations are lifted directly from [Dem65, Proposition 5.4.3]. Let $G$ be a finite Chevalley group over a finite field $\mathbb{F}_q$ with $q$ elements, defined as in Section 2.2.2. Let $R = R^+ \cap R^-$ be as in Section 2.2.1. Let $\alpha, \beta \in R$ such that

$$\beta \neq -\alpha \quad \text{and} \quad |\alpha(H_\beta)| \leq |\beta(H_\alpha)|.$$

Let $l, r \in \mathbb{Z}_{\geq 0}$ be maximal such that

$$\{\beta - l\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \ldots, \beta + r\alpha\} \subseteq R.$$

Note that $l + r \leq 3$ [Hum72, Section 9.4]. Thus, the following analysis includes all the possible $l$ and $r$ values for $\alpha$ and $\beta$.

- $r = 0$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b),$$
- $l = 0$ and $r = 1$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(\pm ab),$$
- $l = 0$ and $r = 2$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(\pm ab)x_{2\alpha+\beta}(\pm a^2 b),$$
- $l = 0$ and $r = 3$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(\pm ab)x_{2\alpha+\beta}(\pm a^2 b)x_{3\alpha+\beta}(\pm a^3 b)x_{3\alpha+2\beta}(\pm a^3 b^2),$$
- $l = 1$ and $r = 1$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(\pm 2ab),$$
- $l = 1$ and $r = 2$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(\pm 2ab)x_{2\alpha+\beta}(\pm 3a^2 b)x_{\alpha+2\beta}(\pm 3ab^2),$$
- $l = 2$ and $r = 1$ implies
$$x_\beta(b)x_\alpha(a) = x_\alpha(a)x_\beta(b)x_{\alpha+\beta}(\pm 3ab),$$

where $a, b \in \mathbb{F}_q$ and $\pm 1$ depends in part on the original choice of Chevalley basis made in Section 2.2.2.

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