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# Math 8174: Homework 3

## Due: October 6, 2010

1. Let $\mathfrak{g}$ be a semi-simple Lie algebra.
   (a) Without using Weyl's Theorem, prove that $\mathfrak{sl}_2(\mathbb{C})$-modules are completely reducible.
   (b) For $\alpha, \beta \in \Phi$, consider the $\mathfrak{sl}_2(\alpha)$-module
   $$V_\alpha^\beta = \mathfrak{g}^{\beta - p\alpha} \oplus \mathfrak{g}^{\beta-(p-1)\alpha} \oplus \cdots \oplus \mathfrak{g}^{\beta+q\alpha}$$
   where $p, q$ are minimal such that $\beta - (p+1)\alpha, \beta + (q+1)\alpha \notin \Phi$.
   Prove that $V_\alpha^\beta$ is irreducible.
2. Let $\mathrm{char}(\mathbb{F}) = p > 0$. Show that the $\mathfrak{sl}_2(\mathbb{F})$-module $V_n$ is irreducible if and only if $n < p$.
3. For $\mathfrak{g} = \mathfrak{o}_n(\mathbb{C})$ or $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, find the angles that occur between the roots.
4. If $R \subseteq E$ is a root system, show that
   $$R^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$$
   is a root system. Then show that there exist two root systems $R$ and $R^\vee$ with bases that are fundamentally different.
   Hint: For the last part work in $\mathbb{R}^n$ for $n \geq 3$.