Math 6350: Homework 2

Due: Friday, September 21

A. Let $S \subseteq \mathbb{C}$ be an open set, and let $f : S \to \mathbb{C}$ be a holomorphic function. Prove the following statements.

(1) If $f'(z) = 0$ for all $z \in S$, then f is constant.

(2) If $f(z) \in \mathbb{R}$ for all $z \in S$, then f is constant.

(3) If $z \mapsto \overline{f(z)}$ is holomorphic, then f is constant.

(4) If $|f(z)|$ is constant, then f is constant.

B. (1) Give a precise definition of a single-valued branch of $\sqrt{1+z} + \sqrt{1-z}$, and prove it is holomorphic.

(2) Prove that $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously holomorphic.

C. Assume f is holomorphic in an open set S with f' continuous and $|f(z) - 1| < 1$ for $z \in S$. Show

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve γ in S.