Math 6250: Homework 6

1. (22.1) Let P_R be finitely generated and projective. Let $T = \mathrm{Tr}_R(P)$. Prove that
   (a) $T^2 = T$, $PT = T$, and $T = \mathrm{Tr}_R(\mathrm{Hom}_R(P, R))$.
   (b) Find isomorphisms $P \otimes_R T \to P$ and $T \otimes_R \mathrm{Hom}_R(P, R) \to \mathrm{Hom}_R(P, R)$.
   (c) If $e \in R$ is an idempotent such that $P \cong eR$, then $T = ReR$.
2. Give the character table for D_{2n} for all n.
3. Given G-modules U and V, the action

$$\begin{array}{ccc} G \times (U \otimes_{\mathbb{F}} V) & \longrightarrow & U \otimes_{\mathbb{F}} V \\ (g, u \otimes v) & \mapsto & (gu) \otimes (gv) \end{array}$$

   makes $U \otimes_{\mathbb{F}} V$ an G-module.
   (a) Find a character formula for $\chi_{U \otimes V}$ in terms of χ_U and χ_V.
   (b) Show that U 1-dimensional implies $U \otimes V$ irreducible if and only if V irreducible.
4. A character $\chi : G \to \mathbb{C}$ is *real valued* if $\chi(G) \subseteq \mathbb{R}$.
   (a) Show with an example that χ_ρ real-valued does not imply that $\rho(G) \subseteq \mathrm{GL}_{\chi_\rho(1)}(\mathbb{R})$.
   (b) Let U and V be irreducible G-modules with χ_U real-valued. Show that $U \otimes_{\mathbb{F}} V$ contains the trivial module if and only if $\chi_U = \chi_V$.