Math 6250: Homework 4

1. (12.2) Prove that if $M = \bigoplus_{i \in \mathcal{I}} M^{(i)}$, complements direct summands, then $M^{(i)}$ is indecomposable for $i \in \mathcal{I}$.

2. (12.3) Consider $\mathbb{R}^{\mathcal{I}}$ for an infinite set $\mathcal{I}$.

   (a) Show that the regular module does not have an indecomposable decomposition.

   (b) For $t \in \mathbb{R}$, let $c_t : \mathcal{I} \to \mathbb{R}$ be given by $c_t(r) = t$ for all $r \in \mathcal{I}$. Show that $\mathbb{R}^{\mathcal{I}}$ has an indecomposable decomposition as a $K = \{c_t \mid t \in \mathbb{R}\}$-module.

   (c) Give an example of an indecomposable K-direct summand of $\mathbb{R}^{\mathcal{I}}$ that is not an $\mathbb{R}^{\mathcal{I}}$-direct summand

3. (12.10) Consider the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$.

   (a) Show that all indecomposible decompositions are equivalent.

   (b) Find a maximal direct summand that is not complemented by the decomposition $\mathbb{Z}(1,1) \oplus \mathbb{Z}(1,2)$.

4. (15.1) Compute the Jacobson radical of each of the following rings:

   (a) $\mathbb{Z}$

   (b) $\mathbb{Z}_n$

   (c) The ring of $n \times n$ upper triangular matrices over a field $\mathbb{F}$.