Math 6250: Homework 2

1. Let C_n be the cyclic group of order n.

   (a) Completely classify the irreducible C_n-modules over the complex numbers $\mathbb{C}$,

   (b) Show how to decompose the regular module $\mathbb{C}C_n$ in terms of irreducibles. That is, find an explicit basis $\mathcal{B} \subseteq \mathbb{C}C_n$ such that for each $b \in \mathcal{B}$, $\mathbb{C}$-span$\{b\}$ is a submodule of $\mathbb{C}C_n$.

2. Let V be a completely reducible A-module over an algebraically closed field $\mathbb{F}$. Prove the converse of Schur's Lemma. That is, if for every A-module homomorphism $\theta : V \to V$ there exists $c \in \mathbb{F}$ such that $\theta(v) = cv$, then V is irreducible.

3. Let A be a semisimple finite dimensional algebra. The *degree sequence* of a A is the sequence $\{\dim(A^\lambda) \mid \lambda \in \hat{A}\}$. Suppose $|\dim(A)| = 20$ and $\text{char}(\mathbb{F})$ does not divide $\dim(A^\lambda)$, $\lambda \in \hat{A}$.

   (a) Show that A has at least one dimension 1 module.

   (b) Show that G has less than 10 possible degree sequences.

   (c) If $A = \mathbb{F}G$, then it turns out that $\dim(\mathbb{F}G^\lambda)$ divides $|G|$. Using this fact, show that in this case $\mathbb{F} \cdot G$ must have at least 4 different one-dimensional modules.

4. (13.4 in book).

   (a) Let I be a proper ideal of a semisimple ring R. Show that R/I is also semisimple.

   (b) Give an example to show that subrings of semisimple rings need not be semisimple.

5. Let A be a semisimple algebra. Suppose $V \subseteq A$ is a submodule of the regular module A.

   (a) Show that there exists an idempotent $e \in A$ (an element such that $e^2 = e$) such that
   $$V = Ae.$$

   (b) Show that if $\theta \in \text{Hom}_A(V, A)$, then
   $$\theta(v) = va, \qquad \text{for some } a \in A.$$

   (c) Show that
   $$\text{End}_A(V) \cong eAe,$$
   as vector spaces.

   Remark: One can show that the two spaces in (c) are in fact anti-isomorphic as algebras.