Math 6140: Homework 8

1. 13.4: 4, 5, 6

2. 13.5: 3, 4, 7

3. Let $\mathbb{F} \subseteq \mathbb{K} \subseteq \mathbb{L}$. Suppose $\alpha \in \mathbb{L}$ is algebraic over $\mathbb{F}$ and let $f = \min_{\alpha,\mathbb{K}}(x)$. Show that the roots in $\mathbb{L}$ and coefficients of f are algebraic over $\mathbb{F}$.

4. Suppose $x^p - 1$ factors completely over a field $\mathbb{F}$ with p prime. Show that for each $a \in \mathbb{F}$, either $x^p - a$ factors completely in $\mathbb{F}[x]$ or is irreducible in $\mathbb{F}[x]$ (Hint: note that the roots of $x^p - a$ all have the same degree).

5. Suppose $\text{char}(\mathbb{F}) = p > 0$, and $\mathbb{K}/\mathbb{F}$ is an algebraic extension. Show that the following are equivalent.

   (a) The only elements in $\mathbb{K}$ that are roots of a separable polynomial in $\mathbb{F}[x]$ are in $\mathbb{F}$.

   (b) If $\alpha \in \mathbb{K}$, then there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\alpha^{p^n} \in \mathbb{F}$.