Math 4140: Homework 1

Due: January 19, 2011

Group actions

1. Let G be a group. Consider functions

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (g,h) & \mapsto & hg \end{array} \quad \text{and} \quad \begin{array}{ccc} G \times G & \longrightarrow & G \\ (g,h) & \mapsto & h^{-1}g \end{array}$$

   Determine whether or not either of these actions give a right action of G on itself (justify your answers).

2. Prove that every finite group G is isomorphic to a subgroup of S_n for some $n \in \mathbb{Z}_{\geq 1}$.

   Hint: First prove that the group of permutations of the set G is isomorphic to $S_{|G|}$, and then use group actions.

3. Consider the right action of G on

$$\mathcal{F}(G) = \{f : G \to \mathbb{C}\}$$

   given by $(f * g)(h) = f(hg^{-1})$, for $f \in \mathcal{F}(G)$, $g, h \in G$.

   (a) Show that this is indeed a group action.

   (b) Under what conditions do the orbits all have the same size?

Vector spaces

1. Show that $\mathcal{F}(G)$ may be viewed as a vector space. What is its dimension? Find an explicit basis for $\mathcal{F}(G)$.

2. Suppose $\phi : U \to V$ is a linear transformation between vector spaces U and V. Suppose $\mathcal{B}_U$ and $\mathcal{B}'_U$ are two bases of U and $\mathcal{B}_V$ and $\mathcal{B}'_V$ are two bases of V. What is the relationship between the matrix a^ϕ corresponding to the basis $\mathcal{B}_U$ and $\mathcal{B}_V$ and the matrix a'^ϕ corresponding to the bases $\mathcal{B}'_U$ and $\mathcal{B}'_V$. Your answer should be in the form of a matrix equation with explicit matrices.

3. Let p be prime and let $\mathbb{Z}_p$ be the integers mod p. How many distinct bases does $\mathbb{Z}_p^n$ have?