Math 4140: Homework 9

Due April 8, 2009

1. Let $\mathfrak{g}$ be the 3-dimensional Lie algebra

$$\mathfrak{g} = \mathbb{C}\text{-span}\{x, y, z\},$$

where

$$[x, y] = z, \qquad [y, z] = x, \qquad [z, x] = y.$$

   (a) Show that $\mathfrak{g}$ is isomorphic to $\mathfrak{o}_3(\mathbb{C})$ (see Problem 4 on the midterm).

   (b) Show that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ in two ways: by an explicit isomorphism and in some other way.

2. Let J be an $n \times n$ matrix with entries in a field $\mathbb{F}$. Define

$$\mathfrak{gl}_n(J, \mathbb{F}) = \{x \in \mathfrak{gl}_n(\mathbb{F}) \mid \text{Transpose}(x)J + Jx = 0\}.$$

   (a) Show that $\mathfrak{gl}_n(J, \mathbb{F})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.

   (b) Find J such that $\mathfrak{gl}_n(J, \mathbb{F}) = \mathfrak{o}_n(\mathbb{F})$.

   (c) Find $\mathfrak{gl}_2(J, \mathbb{R})$, if $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

   (d) Does there exist a matrix J such that $\mathfrak{gl}_2(J, \mathbb{F})$ is the set of all diagonal matrices in $\mathfrak{gl}_2(J, \mathbb{F})$?

Remark. It is worth comparing this problem with Problem 1 on Homework 1.

3. The Heisenberg Lie algebra is the Lie algebra

$$\mathfrak{u}_3 = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}.$$

   (a) Find $\mathfrak{u}_3'$ explicitly.

   (b) Find $Z(\mathfrak{u}_3)$ explicitly.

   (c) Show that $\mathfrak{u}_3$ is the unique 3-dimensional Lie algebra $\mathfrak{g}$ such that $\dim(\mathfrak{g}) = \dim(\mathfrak{u}_3')$ and $\mathfrak{g}' \subseteq Z(\mathfrak{g})$.