Math 4140: Homework 4

Due February 11, 2009

1. Find examples of the following for $V = \mathbb{F}^n$.

   (a) A degenerate symmetric bilinear form.

   (b) A nondegenerate non-symmetric bilinear form.

2. Let $(\cdot,\cdot) : V \times V \to \mathbb{F}$ be a bilinear form. Suppose V has a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. Consider the matrix

$$M = \begin{pmatrix} (v_1, v_1) & (v_1, v_2) & \cdots & (v_1, v_n) \\ (v_2, v_1) & (v_2, v_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & (v_{n-1}, v_n) \\ (v_n, v_1) & \cdots & (v_n, v_{n-1}) & (v_n, v_n) \end{pmatrix}.$$

   (a) What is M when $V = \mathbb{F}^n$, with the natural basis, and $(\cdot,\cdot)$ is the dot product?

   (b) Show that in general

$$\left(\sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j v_j \right) = (a_1, a_2, \ldots, a_n) M \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

   (c) Show that $(\cdot,\cdot)$ is nondegenerate if and only if M is invertible.

   Hint: Show that the column vectors of M are linearly independent if and only if $(\cdot,\cdot)$ is nondegenerate.

3. Let $(\cdot,\cdot) : V \times V \to \mathbb{F}$ be a symmetric bilinear form. Suppose V has a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. Show that $(\cdot,\cdot)$ is nondegenerate if and only if there exists a basis $\mathcal{B}' = \{v_1', v_2', \ldots, v_n'\}$ such that $(v_i, v_j') = \delta_{ij}$.

   Hint: For the hard direction, combine the facts that $V \to V^*$ $(v \mapsto (\cdot, v))$ is an isomorphism if $(\cdot,\cdot)$ is nondegenerate, and that V^* has a particularly nice natural basis.