Math 3170: Homework 10

Due: November 14, 2012

1. How many spanning trees are there of the complete graph K_n that have no vertex with degree greater than 2.

2. The *distance* $d(u,v)$ between two vertices u and v in a connected graph is the smallest number of edges needed to construct a path between u and v. The *center* of a connected graph G is the set

$$\{v \in V_G \mid \sum_{u \in V_G} d(u,v) \text{ is minimal}\}.$$

   Prove that if T is a tree, then the center of T is either a vertex or a pair of adjacent vertices.

3. Suppose a tree T has exactly one vertex of degree i for all $2 \leq i \leq m$ (all other vertices have degree 1). How many vertices does T have?

4. Let G be a connected simple graph, and let S and T be spanning trees of G.

   (a) Show that if $e \in E_S$, then there exists $f \in E_T$ such that the tree S' obtained by deleting e and adding f is a spanning tree of G.

   (b) Show that there is a sequence of spanning trees

$$S = T_0, T_1, \ldots, T_\ell = T$$

   such that T_i is obtained from T_{i-1} be removing an edge and adding another.

5. Let G_n be obtained from K_n by removing an edge. Find and prove a formula for the number of spanning trees of G_n.

   Hint: Count the number of spanning trees of K_n that use the deleted edge.