Math 3170: Homework 3

Due: September 15, 2010

1. Prove
$$3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$
in two different ways.

2. Prove
$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$
in two different ways.

3. What is the coefficient of x^n in the power series $\sqrt[3]{1-2x}$?

4. (a) What is the power series for
$$\frac{1}{(1-x)^2}?$$
   (b) Compute the first ten terms of the sequence
$$h_n = h_{n-1} + h_{n-2} - h_{n-3},$$
   with $h_0 = 0$, $h_1 = 1$, $h_2 = 1$.
   (c) Use generating functions to find a closed formula for h_n (although it is easy to do so without generating functions).

5. Say a sequence $a_1, a_2, \ldots, a_{2n}$ of n ones and n minus ones is *good* if for each $1 \le k \le 2n$, the sum $a_1 + a_2 + \cdots + a_k \ge 0$. Let
$$\mathrm{se}_n = \#\{\text{good sequences of length } 2n\}.$$
For example,
$$\begin{aligned} \mathrm{se}_3 &= \#\left\{ \begin{array}{c} (1,-1,1,-1,1,-1),(1,1,-1,-1,1,-1),(1,-1,1,1,-1,-1), \\ (1,1,1,-1,-1,-1),(1,1,-1,1,-1,-1) \end{array} \right\} \\ &= 5. \end{aligned}$$
Show that se_n is the nth Catalan number by constructing a bijection between Dyck paths and good sequences.