1 Finishing the proof of the Catalan Theorem

Date: September 13, 2010

Difference between math k and normal k. Larger underscore uses braces $u_{c,k}^{5200}$

Theorem 1.1. *The kth Catalan number c_k is the number of Dyck paths from $(0,0)$ to $(0,2k)$.*

Proof. Last time we defined

$$\mathcal{C}_n = \left\{ \begin{array}{c} \text{Dyck paths from} \\ (0,0) \text{ to } (0,2k) \end{array} \right\}$$

and

$$\mathcal{T}_n = \left\{ \begin{array}{c} \text{Paths from} \\ (0,0) \text{ to } (0,2k) \text{ using} \\ n\ (1,1)\text{-steps and } n\ (1,-1)\text{-steps} \end{array} \right\}$$

and

$$\mathcal{B}_n = \left\{ \begin{array}{c} \text{Paths passing under } x\text{-axis from} \\ (0,0) \text{ to } (0,2k) \text{ using} \\ n\ (1,1)\text{-steps and } n\ (1,-1)\text{-steps} \end{array} \right\},$$

so that

$$\mathcal{T}_n = \mathcal{C}_n \cup \mathcal{B}_n.$$

We also showed that

$$\mathcal{B}_n \longleftrightarrow \mathcal{D}_n = \left\{ \begin{array}{c} \text{Paths from} \\ (0,0) \text{ using } n+1\ (1,1)\text{-steps} \\ \text{and } n-1\ (1,-1)\text{-steps} \end{array} \right\},$$

so

$$\begin{aligned}
|\mathcal{C}_n| &= |\mathcal{T}_n| - |\mathcal{D}_n| && \text{(BLAH)} \\
&= \binom{2n}{n} - \binom{2n}{n+1} && (1) \\
&= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} && (2) \\
&= \frac{(n+1)(2n)!}{(n+1)n!n!} - \frac{n(2n)!}{n(n+1)!(n-1)!} && (3) \\
&= \frac{1}{n+1}\frac{(2n)!}{n!n!} && (4) \\
&= \frac{1}{n+1}\binom{2n}{n}. && (5)
\end{aligned}$$

Because we have shown that (BLAH) is equal to (5), the theorem now follows. □

2 Appendix

2.1 Binomial Theorem

Theorem 2.1 (Binomial Theorem)**.**

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

2.2 Greek symbols

α, β, γ, Γ, Δ, χ, ϵ, ε

2.3 Sets

An elements $x \in S$. A set $R \subseteq S$. Note that $a \leq b$ and $a \geq b$, so $a = b$.

2.4 Matrices

Consider the matrix

$$\begin{pmatrix} 1 & x^2 & \sqrt{x^2 - 1} & 0 \\ 0 & 1 & \lim\limits_{n \to \infty} \frac{1}{n} & \sqrt[3]{42} \\ 0 & 0 & \int_0^5 x^2 dx & 3 \end{pmatrix}$$

Let's define

$$S = \{n \in \mathbb{Z} \mid 2 \text{ divides } n - 1\}.$$

To say "implies," write implies. $\implies$