# Math 3140: Homework 10

## Due: Wednesday, November 19

A.

16.11. (Fourth Isomorphism Theorem) Let $\varphi : G \to H$ be a surjective homomorphism with kernel $K$.

(a) For every subgroup $B \subseteq H$, show that the set

$$\varphi^{-1}(B) = \{g \in G \mid \varphi(g) \in B\}$$

is a subgroup of $G$ that contains $K$.

(b) Show that there is a bijection between

$$\left\{ \begin{array}{c} \text{Subgroups } A \subseteq G \\ \text{such that } K \subseteq A \end{array} \right\} \longleftrightarrow \{ \text{ Subgroups } B \subseteq H \ \}.$$

16.12 An *maximal* normal subgroup $N$ of $G$ is a normal subgroup such that if $H \supseteq N$ is a normal subgroup of $G$, then $H = N$ or $H = G$. Show that $N$ is a maximal normal subgroup of $G$ if and only if $G/N$ is simple.

B.

17.1. Let $G = \langle (1, 2, 3)(4, 5), (7, 8)\rangle \subseteq S_8$. Then $G$ acts on the set $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Calculate the orbits of the $G$-action in $X$, and the stabilizers of 1, 2 and 7.

17.4 Let $G$ act on a set $X$, and let $O$ be an orbit in $G$. Show that $\mathrm{Stab}_G(x) = \mathrm{Stab}_G(y)$ for all $x, y \in O$ if and only if $\mathrm{Stab}_G(x) \lhd G$.

17.7-8. **The diagonal action.** Suppose $G$ acts on $X$ and $Y$. Let

$$\begin{array}{ccc} G \times (X \times Y) & \longrightarrow & X \times Y \\ (g, (x, y)) & \mapsto & (g(x), g(y)) \end{array} \qquad (*)$$

i. Show that $(*)$ gives an action of $G$ on $X \times Y$.

ii. For example, find the orbits of $G = \langle (1, 2, 3, 4), (2, 4)\rangle \subseteq S_4$ acting on $X \times X$ diagonally, where $X = \{1, 2, 3, 4\}$. Is this action transitive?

iii. In general, find the stabilizer of $(x, y) \in X \times Y$ (in terms of the stablizers of $x$ and $y$).

17.10. The *centralizer* of $g \in G$ is the subgroup

$$C_G(g) = \{h \in G \mid hgh^{-1} = g\}.$$

i. Identify a set $X$ with an action of $G$ on that set such that $C_G(g) = \mathrm{Stab}_G(g)$ for that action.

ii. Show that if some conjugacy class of $G$ has exactly two elements, then $G$ is not simple.