Math 3140: Homework 9

Due: Wednesday, November 6

A. (a) Suppose R and S are rings. Give a careful definition of what you think a *ring homomorphism* should be.

(b) Let $\varphi : R \to S$ be a ring homomorphism. Define the *kernel* of φ to be the set

$$\ker(\varphi) = \{r \in R \mid \varphi(r) = 0\}.$$

Show that for all $r \in R$ and $k \in \ker(\varphi)$,

$$rk, kr \in \ker(\varphi).$$

(c) An *ideal* of a ring R is a subring I such that

$$ri, ir \in I \qquad \text{for all } i \in I,\ r \in R.$$

Find a multiplicative function and an additive function that makes

$$R + I = \{r + I \mid r \in R\}$$

a ring.

(d) Show that $I \subseteq R$ is an ideal if and only if there exists a ring homomorphism $\varphi : R \to S$ for some ring S such that $I = \ker(\varphi)$.

B. Find an example of a homomorphism that is neither injective nor surjective.

C. 15.2 Find all normal subgroups of D_n.

15.7 Let $K \lhd G \times H$ be such that

$$K \cap (\{1_G\} \times H) = \{(1_G, 1_H)\} = K \cap (G \times \{1_H\}).$$

Show that K must be abelian.

15.12 Find a proper normal subgroup of A_4. Show that any non-trivial normal subgroup H of A_5 must contain a 3-cycle, and use 14.5 to conclude that $H = A_5$, thereby proving A_5 is simple.