Math 2001: PHW8

1. Consider the following

   Claim. *The number $n(n+1)$ is an odd number for every n.*

   Proof. Assume the statement is true for n. We prove the statement for $n+1$ by induction. Note that

   $$(n+1)((n+1)+1) = n(n+1) + 2(n+1).$$

   By induction $n(n+1)$ is odd. Thus, $(n+1)((n+1)+1)$ is the sum of an odd number $n(n+1)$ and an even number $2(n+1)$. The sum of an odd number and an even number is odd. Thus, we have proved the claim by induction. □

   I checked the claim and it doesn't seem to work for $n = 15$, since $15 \cdot 16 = 240$, which is even. What is wrong with the proof?

2. For each of the following sequences,

   - Give a formula for the nth term in the sequence,
   - Give a recursive definition for the sequence (ie. initial values and a recursive equation).

   (a) $\{1, 2, 3, 4, 5, \ldots\}$

   (b) $\{1, 2, 4, 8, 16, 32, \ldots\}$

   (c) $\{1, 2, 6, 24, 120, \ldots\}$

3. Let $f_0, f_1, \ldots$ be the Fibonacci sequence. For each of the following

   - Decide whether the identity is easier to prove by induction or directly using Binet's formula (and some algebra). Explain.
   - Prove the identity using your preferred method.

   (a) $\displaystyle\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$.

   (b) $\displaystyle\sum_{k=0}^{n} f_k = f_{n+2} - 1$.

   (c) $f_{2n+1} = f_{n+1}^2 + f_n^2$.

4. The Lucas sequence is given by

   $$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2}, \; n \geq 3.$$

   (a) Find the first 6 values of the Lucas sequence.

   (b) What should L_0 be defined to be to not mess up the recursion?

   (c) Use induction to prove that

   $$L_n = f_{n-1} + f_{n+1}, \qquad \text{for } n \geq 1,$$

   where f_n is the nth Fibonacci number.

(d) Prove that

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$