Math 2001: Homework 6

Due: October 15, 2008

Give complete justifications for all your answers.

Problem 1

1. Prove that the product of two even numbers is always even.
2. Consider the set

$$\begin{aligned} A &= \{4n+1 \in \mathbb{Z} \mid n \in \mathbb{Z}, n \geq 0\} \\ &= \{1, 5, 9, 13, \ldots\} \end{aligned}$$

   Show that the product of any two elements in A is another element in A.

3. Consider two pairs of integers $(1597, 987)$ and $(1590, 997)$.
   - Find $\gcd(1597, 987)$ and $\gcd(1590, 997)$ using the Euclidean algorithm.
   - Which pair takes more steps in the Euclidean algorithm? Give an explanation for why this might be?
   - For the faster pair (m, n), find $k, l \in \mathbb{Z}$ so that $\gcd(m, n) = km + ln$.

Problem 2

Let $F_0, F_1, \ldots$ be the Fibonacci sequence. For each of the following

- Decide whether the identity is easier to prove by induction or directly using Binet's formula (and some algebra). Explain.
- Prove the identity using your preferred method.

1. $\sum_{k=0}^{n} F_k = F_{n+2} - 1$.
2. $F_{2n+1} = F_{n+1}^2 + F_{n-1}^2$.
3. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$.

Problem 3

The purpose of this problem is to prove the assertion that for positive integer m and n,

$$mn = \gcd(m, n)\operatorname{lcm}(m, n).$$

(a) Describe the prime factorization of $\gcd(m, n)$ in terms of the prime factorization of m and the prime factorization of n.

(b) Describe the prime factorization of $\operatorname{lcm}(m, n)$ in terms of the prime factorization of m and the prime factorization of n.

(c) Combine (a) and (b) to prove the result.