Testing for a Semilattice Term

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- *There is a straightforward, but inefficient algorithm to settle this question: Compute the free algebra in $\mathbf{V}(\mathbf{A})$ generated by $\{x, y\}$ and look for a binary term that satisfies these equations.*
  As a function of $|A|$, the run time of this algorithm grows exponentially.

Is there a better way?

Theorem (Freese-Val.)

Let $\mathbf{A}$ *be a finite algebra. The problem of deciding if a finite algebra* $\mathbf{A}$ *has a semilattice term is EXP-TIME complete.*

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- *We provide a way to construct a finite algebra $\mathbf{A}_I$ from an instance I such that:*
  - *If I is a* **yes** *instance, then $\mathbf{A}_I$ has a semilattice term, and*
  - *If I is a* **no** *instance, then $\mathbf{A}_I$ has no non-trivial idempotent terms.*

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- An algebra is idempotent if all of its basic operations are idempotent.

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Conjecture (Kazda-Val.)

For Σ *a idempotent, linear, strong Maltsev condition, there is a polynomial-time test to determine if a finite idempotent algebra generates a variety that satisfies* Σ.

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- *for every proper subset S of A_n there is a binary term operation of $\mathbf{A}_n$ whose restriction to S satisfies the semilattice identities, and*
- *$\mathbf{A}_n$ does not have a semilattice term.*

Description of $\mathbf{A}_n$

local semilattice terms

With $A_n = \{0, 1, 2, \ldots, n-1\}$, and $i \in A_n$, let $b_i(x, y)$ equal the minimum of x and y, with respect to the ordering:

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except that $b_i(i, i-1) = i-1$.

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- *it is not a semilattice operation on A_n.*
- *It can be shown that $\mathbf{A}_n$ has no semilattice term in spite of this.*

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- What about in the idempotent case?

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Theorem

There is a polynomial-time test to determine if a given finite idempotent algebra $\mathbf{A}$ *has a flat semilattice term operation. In fact,* $\mathbf{A}$ *has a flat semilattice term operation if and only if*
for all a, b, $c \neq d \in A$, *there is a term operation* $t(x, y)$ *such that*

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Remark

So, to test if a finite idempotent algebra has a flat semilattice term operation, we need to show that for all a, b, $c \neq d \in A$, *the tuple* $(0, 0, 0)$ *is in the subalgebra of* $\mathbf{A}^3$ *generated by* $\{(a, 0, c), (0, b, d)\}$.

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Theorem

The problem of deciding if a finite idempotent algebra $\mathbf{A}$*, along with a distinguished element* 1*, has a bounded semilattice term operation with maximum element* 1 *is EXP-TIME complete.*

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- To establish hardness, we present a procedure for building a finite idempotent algebra $\mathbf{A}_I$ from an instance $I = (A, \mathcal{F}, h(x))$ of GEN-CLO′.

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- The universe of $\mathbf{A}_I$, A_I, consists of A and two new elements 0 and 1 that will serve as the smallest and largest elements of the semilattice that will arise if I is a **yes** instance.

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- The universe of $\mathbf{A}_I$, A_I, consists of A and two new elements 0 and 1 that will serve as the smallest and largest elements of the semilattice that will arise if I is a **yes** instance.
- Each function $g : A^k \to A$ can be expanded to an idempotent operation g' on A_I in a natural way as follows:

$$g'(x_1, \dots, x_k, y) = \begin{cases} g(x_1, \dots, x_k) & \text{if } \{x_1, \dots, x_k\} \subseteq A \text{ and } y = 1; \\ y & \text{if } x_i = y \text{ for all } 1 \leq i \leq k; \\ 0 & \text{otherwise.} \end{cases}$$

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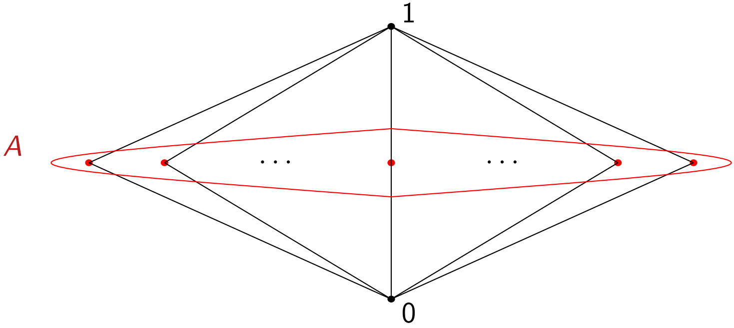
- $\mathbf{A}_I$ is the algebra with universe $A \cup \{0, 1\}$ and with basic operations f' for each $f \in \mathcal{F}$, plus,



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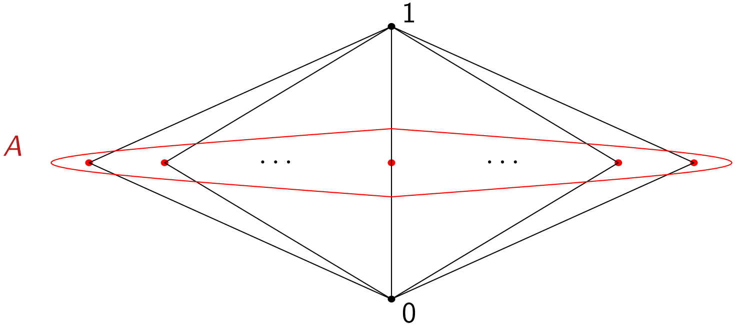

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- To prove the main result, that testing for a semilattice term operation is hard for idempotent algebras, we reduce the bounded semilattice problem to this one.

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- if not, then $\mathbf{A}^{\diamond}$ will not have any semilattice operation.

Theorem

The problem of deciding if a finite idempotent algebra has a semilattice term is EXP-TIME complete.

Beyond semilattice terms

Question

Is it the case that testing for any non-linear, strong, idempotent Maltsev condition is EXP-TIME hard, even for idempotent algebras?

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2-semilattices

A natural example to consider is that of having a 2-semilattice term, i.e., a binary term $x \wedge y$ that satisfies the equations

$$x \wedge x \approx x, \quad x \wedge y \approx y \wedge x, \quad x \wedge (x \wedge y) \approx x \wedge y.$$