Ultralocal term operations

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The proof relies on two earlier papers by Vaggione, one on sheaf representations in congruence distributive varieties, and the other on the definability of functions by semantical conditions.

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$\Gamma_\infty(\mathbf{A}) = \bigcap_\kappa \Gamma_\kappa(\mathbf{A})$ = the clone of term operations of $\mathbf{A} = \Gamma_{\omega+|A|^+}(\mathbf{A})$

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$\Gamma_{\omega^+}(\mathbf{A}) = \Gamma_\infty(\mathbf{A})$ = the clone of $\mathbf{A}$.

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If $\mathbf{A}$ *has a* d*-ary near unanimity operation, then* $\Gamma^*_d(\mathbf{A}) = \Gamma^*_\infty(\mathbf{A}) = \Gamma_\infty(\mathbf{A})$.

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Here $\Gamma^*_\kappa(\mathbf{A})$ is the set of F such that $F_{\mathcal{U}} \in \Gamma^*_\kappa(\mathbf{A}_{\mathcal{U}})$ for all $\mathcal{U}$.

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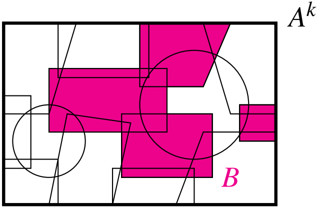

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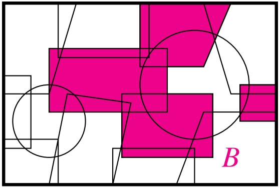

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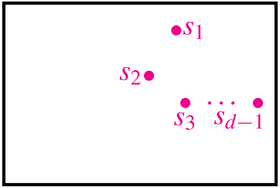

$f(s_i) = t(s_i)$ for $s_i \in S$

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A clone that is not ultralocally closed

Example If $\mathbf{G} := (\omega; \mathrm{Alt}_\omega)$, then for $A = \omega$

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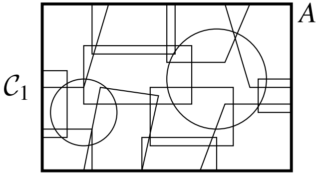
$\Rightarrow$: $\Leftarrow$: Enough: $F = (u\,v)$ is an ul.term op.



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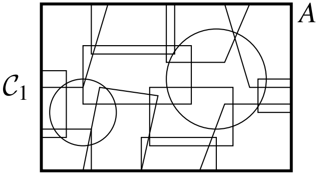

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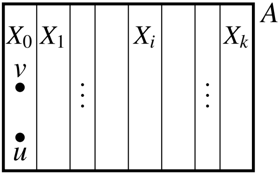

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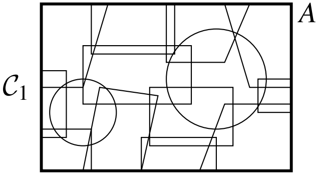

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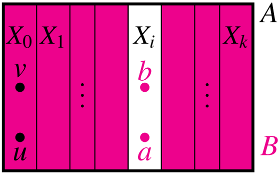

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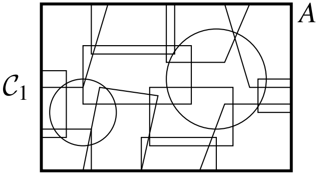

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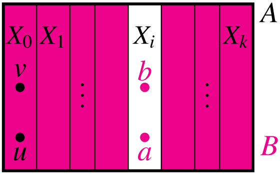

- $f|_B = (f \circ (a\,b))|_B$ if $X_0 \subseteq B$

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