

# How Aliens Do Math

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Why is there Arithmetic?

## Why is there Arithmetic?

- Perception

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- Language

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- Perception
- Language
- Perspective

## Easy problems

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- In  $\mathbb{C}$ , solve  $x^5 + 3x + 1 = 0$ .

## Algebras

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## Natural numbers

$\mathbb{N} = \langle \mathbb{N}, +, \times, 0, 1 \rangle$

## Infinite cardinals

$\aleph_0, \aleph_1, \aleph_2, \dots$  with

$$x + y = \max(x, y)$$

$$x \times y = \max(x, y)$$

Euclid

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Fundamental Theorem of Arithmetic

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Duh!

If  $x \times y = 0$ , then  $x = 0$  or  $y = 0$ .

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## Advantage

Algebra and logic (using XOR) are the same!

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- How many prime numbers on Venus?



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- What is the Fundamental Theorem of Venutian Arithmetic?

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- Solve  $2x = 1$  on Venus
- Solve  $x^2 = 2$  on Venus

## Questions

- How many prime numbers on Venus?
- What is the Fundamental Theorem of Venutian Arithmetic?
- When is  $x \times y = 0$ ?



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## Problem

Solve  $x^2 = 2$  on Mars

**Use Martian complex numbers**  $a + bu$  where  $u^2 = 2$

On Jupiter, everything is BIG!

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## Arithmetic

0, 1, 2, 3, ... with

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- $x + (y + z) = (x + y) + z$

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## Semilattices

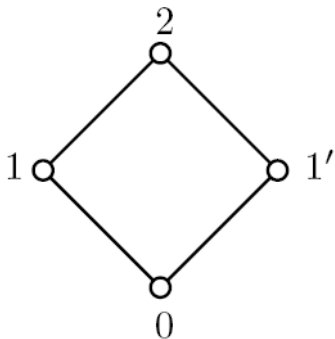
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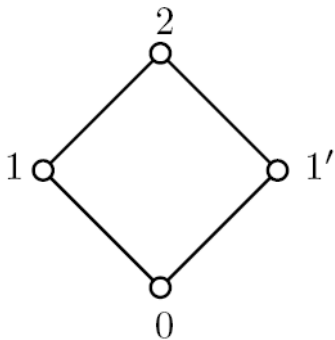
## Order

$x \leq y$  if and only if  $x + y = y$

# Counting to Two on Saturn



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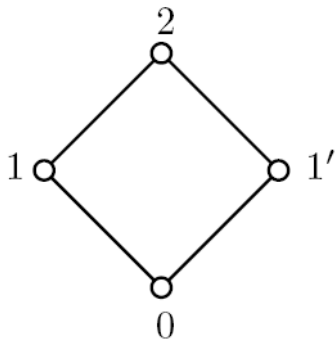


## Questions

- $1 + 1 = ?$



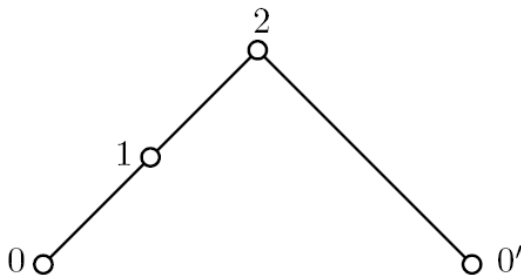
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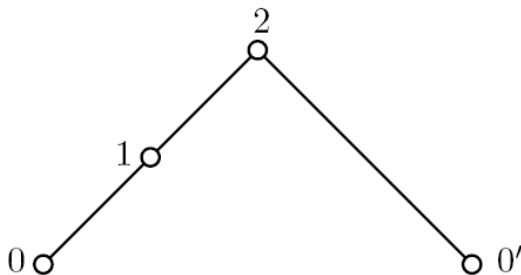
## Questions

- $1 + 1 = ?$
- $1 + 1' = ?$

# Counting to Two on Uranus



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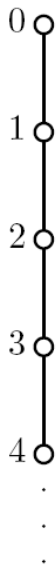


Question

$$0 + 0' = ?$$

# Counting to Two on Neptune

0  
1  
2  
3  
4  
.  
.  
.





## Arithmetic

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$\mathbb{T} = \langle T, +, \times \rangle$  satisfying

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- $x + y = x$



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- Lots of nice properties

## Arithmetic

$\mathbb{T} = \langle T, +, \times \rangle$  satisfying

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- Lots of nice properties
- Easy to solve equations

## Arithmetic

- $x + y = x$
- $x \times y = y$

## Easy problems on Pluto

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## Arithmetic

- $x + y = x$
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- Solve  $x + 7 = 4$ .
- Solve  $7 + x = 4$ .

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- $x + y = x$
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- Solve  $x + 7 = 4$ .
- Solve  $7 + x = 4$ .
- Solve  $5x = 2$ .

## Arithmetic

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- Solve  $x + 7 = 4$ .
- Solve  $7 + x = 4$ .
- Solve  $5x = 2$ .
- Solve  $x5 = 2$ .





## Arithmetic

*	R	P	S
R	R	P	R
P	P	P	S
S	R	S	S

??????????

??????????

# MAHALO!