Promise Constraint Satisfaction Problems

Athena Sparks

CU Boulder

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Mathematics
UNIVERSITY OF COLORADO BOULDER

The constraint satisfaction problem (CSP)

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Definition

Let Γ be a finite relational language and $\mathbb{A}$ a Γ-structure.

CSP($\mathbb{A}$): Input: Φ a primitive positive Γ-sentence
Output: True, if $\mathbb{A} \models \Phi$
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Feder–Vardi Dichotomy Conjecture

$\mathrm{CSP}(\mathbb{A})$ is either in P or NP-complete.

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Notes

1. Since there is a homomorphism from $\mathbb{A} \to \mathbb{A}'$, if $\mathbb{A} \models \Phi$, then $\mathbb{A}' \models \Phi$.
2. $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}) = \mathrm{CSP}(\mathbb{A})$.

Example: k-colorability

Let K_k be the complete graph on k vertices.

$$G = \langle [n]; E \rangle \text{ is } k\text{-colorable} \quad \Leftrightarrow \quad K_k \models \exists x_1 \cdots x_n \bigwedge_{(i,j) \in E} x_i \neq x_j .$$

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True, if G is k-colorable

False, if G is not k-colorable

$\mathrm{PCSP}(K_k, K_n)$ for $k \leq n$

True, if G is k-colorable

False, if G is not n-colorable

Algebraic Approach to CSPs

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Polymorphism Clone

Let $\mathbb{A}$ be a relational structure.

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1. $\mathrm{Pol}(\mathbb{A})$ is closed under composition and contains all projection maps.
2. $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$, then $\mathrm{CSP}(\mathbb{B}) \leq_p \mathrm{CSP}(\mathbb{A})$.

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Theorem (Bulatov, Zhuk 2017)

Let $\mathbb{A}$ be a finite relational structure with all constant relations. Then $\mathrm{CSP}(\mathbb{A})$ is in P if $\mathbb{A}$ has a weak near-unanimity (WNU) polymorphism, and $\mathrm{CSP}(\mathbb{A})$ is NP-complete otherwise.

Schaefer's Theorem (1978)

Let $\mathbb{A}$ be a relational structure over a two element domain. If $\mathrm{Pol}(\mathbb{A})$ contains one of the following:

- constant unary opertation 0
- constant unary opertation 1
- binary max
- binary min
- ternary majority
- ternary minority

then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time. Otherwise, $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

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Given an instance Φ of $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$, then Φ is an instance of $\mathrm{CSP}(\mathbb{A})$ and of $\mathrm{CSP}(\mathbb{A}')$. Either decides the $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$.

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Sandwich Lemma

Let $\mathbb{A}, \mathbb{A}', \mathbb{B}$ be Γ-structures and $h : \mathbb{A} \to \mathbb{A}'$ a homomorphism such that h factors through $\mathbb{B}$. Then $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}') \leq_p \mathrm{CSP}(\mathbb{B})$.

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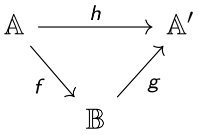
Proof.



Let Φ be an instance of $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$.

- If $\mathbb{A} \models \Phi$, then $\mathbb{B} \models \Phi$.
- If $\mathbb{A}' \not\models \Phi$, then $\mathbb{B} \not\models \Phi$.

□

Example

Let $\mathbb{A}$, $\mathbb{A}'$, $\mathbb{B}$ be Boolean structures with a single 4-ary relation R with the following interpretations:

$$R^{\mathbb{A}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad R^{\mathbb{A}'} = \{0,1\}^4 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$R^{\mathbb{B}} = \{\overline{x} \in \{0,1\}^4 \ : |\overline{x}| \text{ is odd}\}$.

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Note:

- $\mathrm{CSP}(\mathbb{A}), \mathrm{CSP}(\mathbb{A}')$ are NP-complete
- $\mathrm{CSP}(\mathbb{B})$ is in P (has ternary minority polymorphism)
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By the Sandwich Lemma, $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}') \leq_p \mathrm{CSP}(\mathbb{B}) \in \mathrm{P}$.

Algebraic Approach to PCSPs

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Polymorphisms

Let $\mathbb{A}, \mathbb{A}'$ be relational structures over the same signature.

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$\mathrm{Pol}(\mathbb{A}, \mathbb{A}')$ is a clonoid

$\mathrm{Pol}(\mathbb{A}, \mathbb{A}')$ closed under taking minors. For $f\colon A^k \to B$ and $\sigma\colon [k] \to [n]$, $f^\sigma(x_1, \ldots, x_n) := f\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right)$ is a *minor* of f.

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Note

$\mathrm{Pol}(\mathbb{A}, \mathbb{A}')$ is not closed under composition and does not contain projections.

PP-Definability

Let $\mathbb{A}, \mathbb{A}'$ be Γ-structures. A pair $(P, Q) \in \mathcal{P}(A^n) \times \mathcal{P}((A')^n)$ is **pp-definable** from $(\mathbb{A}, \mathbb{A}')$ if there exists a pp-formula $\exists \overline{y}\, \psi(\overline{x}, \overline{y})$ over Γ such that

- if $\overline{a} \in P$, then $\mathbb{A} \models \exists \overline{y}\, \psi(\overline{a}, \overline{y})$, and
- if $\mathbb{A}' \models \exists \overline{y}\, \psi(\overline{b}, \overline{y})$, then $\overline{b} \in Q$.

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Galois Correspondence

$(\mathbb{B}, \mathbb{B}')$ is pp-definable from $(\mathbb{A}, \mathbb{A}')$ if and only if $\mathrm{Pol}(\mathbb{A}, \mathbb{A}') \subseteq \mathrm{Pol}(\mathbb{B}, \mathbb{B}'$.

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Theorem (Brakensiek, Guruswami, 2017)

If $\mathrm{Pol}(\mathbb{A}, \mathbb{A}') \subseteq \mathrm{Pol}(\mathbb{B}, \mathbb{B}')$, then $\mathrm{PCSP}(\mathbb{B}, \mathbb{B}') \leq_p \mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$.

Folded Symmetric Boolean Case (Brakensiek, Guruswami)

Let Γ be a finite relational language, $\mathbb{A}, \mathbb{A}'$ be Boolean Γ-structures, and $h : \mathbb{A} \to \mathbb{A}'$ a homomorphism.

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Important Functions

- $\textsc{Zero}_k(x) = 0$
- $\textsc{One}_k(x) = 1$
- $\textsc{And}_k(x) = \bigwedge_{i=1}^k x_i$
- $\textsc{Or}_k(x) = \bigvee_{i=1}^k x_i$

For k odd:

- $\textsc{Par}_k(x) = \bigoplus_{i=1}^k x_i$
- $\textsc{Maj}_k(x) = \begin{cases} 1 & \sum_{i=1}^k x_i > k/2 \\ 0 & \text{otherwise} \end{cases}$
- $\textsc{AT}_k(x) = \begin{cases} 1 & \sum_{i=1}^k (-1)^{i-1} x_i > 0 \\ 0 & \text{otherwise} \end{cases}$

Folded Symmetric Boolean Case (Brakensiek, Guruswami)

Lemma

If $\mathrm{Pol}(\mathbb{A}, \mathbb{A}')$ satisfies any one of the following:

1. contains $\textsc{Zero}_k$, $\textsc{One}_k$, $\textsc{And}_k$, $\textsc{Or}_k$, $\overline{\textsc{Zero}}_k$, $\overline{\textsc{One}}_k$, $\overline{\textsc{And}}_k$, or $\overline{\textsc{Or}}_k$ for all k
2. contains $\textsc{Par}_k$, $\textsc{Maj}_k$, AT_k, $\overline{\textsc{Par}}_k$, $\overline{\textsc{Maj}}_k$, or $\overline{\mathrm{AT}}_k$ for all k odd

then $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$ is in P.

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Definitions

A function $f : \{0,1\}^k \to \{0,1\}$ is **folded** if $f(\neg x) = \neg f(x)$ for all $x \in \{0,1\}^k$.

A k-ary relation R is **symmetric** if for all $x \in R$ and permutations $\sigma : [k] \to [k]$, we have $(x_{\sigma(a)}, \ldots, x_{\sigma(k)})$.

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Theorem

If $\mathbb{A}, \mathbb{A}'$ are symmetric and $\mathrm{Pol}(\mathbb{A}, \mathbb{A}')$ is folded, then if at least one of PAR_k, MAJ_k, AT_k, $\overline{\mathrm{PAR}}_k$, $\overline{\mathrm{MAJ}}_k$, or $\overline{\mathrm{AT}}_k$ is in $\mathrm{Pol}(\mathbb{A}, \mathbb{A})$ for all k odd, then $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$ is in P. Otherwise, it is NP-hard.

Proof Idea

For hardness use reduction from GapLabelCover.

More on Clonoids

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Clonoid homomorphisms (cf. Barto, Opršal, Pinsker, 2017)

Let $\mathcal{A}$ and $\mathcal{B}$ be clonoids. A **clonoid homomorphism** $\varphi : \mathcal{A} \to \mathcal{B}$ is a mapping that

1. preserves arities
2. commutes with minors, that is $\varphi(f^\sigma) = (\varphi(f))^\sigma$ for any $f \in \mathcal{A}$ and $\sigma : [k] \to [n]$ where k is the arity of f and $n \in \mathbb{N}$.

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Note

(2) is equivalent to preserving identities of height 1, i.e.

$$\varphi(f(\pi^n_{i_1}, \ldots, \pi^n_{i_k})) = \varphi(f)(\pi^n_{i_1}, \ldots, \pi^n_{i_k})$$

for all $k, n \geq 1$, all $i_1, \ldots, i_k \in [n]$, and any k-ary operation $f \in \mathcal{A}$.

Let $\mathcal{A}$ be a clonoid with domain A and codomain A'.

Reflections of $\mathcal{A}$, $\mathrm{R}(\mathcal{A})$

All clonoids $\mathcal{B}$ obtained as follows:

Given $h_1 : B \to A$, $h_2 : A' \to B'$, and $f \in \mathcal{A}$, say k-ary

$$\begin{array}{ccc} A^k & \xrightarrow{f} & A' \\ {\scriptstyle h_1}\uparrow & & \downarrow{\scriptstyle h_2} \\ B^k & \xrightarrow[g_f]{} & B' \end{array}$$

Define $g_f : B^k \to B'$

$$(x_1, \ldots, x_k) \mapsto h_2(f(h_1(x_1), \ldots, h_1(x_k)))$$

$$\mathcal{B} := \{g_f \mid f \in \mathcal{A}\}$$

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Extensions of $\mathcal{A}$, $\mathrm{E}(\mathcal{A})$

All clonoids $\mathcal{B} \supseteq \mathcal{A}$.

Tying it all together

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Theorem

Let $\mathbb{A}, \mathbb{A}'$ be Γ-structures, $\mathbb{B}, \mathbb{B}'$ be Δ-structures, and $\mathcal{A} = \mathrm{Pol}(\mathbb{A}, \mathbb{A}')$ and $\mathcal{B} = \mathrm{Pol}(\mathbb{B}, \mathbb{B}')$. Then $\mathcal{B} \in \mathrm{ERP}(\mathcal{A})$ if and only if there exists a clonoid homomorphism $\mathcal{A} \to \mathcal{B}$.

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Corollary

If either of the above hold, then $\mathrm{PCSP}(\mathbb{B}, \mathbb{B}') \leq_p \mathrm{PCSP}(\mathbb{A}, \mathbb{A}')$.

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- What are the clonoids on $\{0, 1\}$?
- Is there a more general concept than clonoid homomorphisms that gives polytime reductions between PCSPs?