Higher Dimensional Congruences

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Overview of Talk

1. Motivation from Commutator Theory

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2. Higher Dimensional Congruence Relations

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3. A Stronger Term Condition and Commutator

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Definition (Term Condition)

Let $\mathbb{A}$ be an algebra and take $\alpha, \beta, \delta \in \mathrm{Con}(\mathbb{A})$. We say that α **centralizes** β **modulo** δ when the following condition is met:

- For all $t \in \mathrm{Pol}(\mathbb{A})$ and $\mathbf{a}_0 \equiv_\alpha \mathbf{b}_0$ and $\mathbf{a}_1 \equiv_\beta \mathbf{b}_1$ with $|\mathbf{a}_0| + |\mathbf{a}_1| = \sigma(t)$,

$$\Big(t(\mathbf{a}_0, \mathbf{a}_1) \equiv_\delta t(\mathbf{a}_0, \mathbf{b}_1) \implies t(\mathbf{b}_0, \mathbf{a}_0) \equiv_\delta t(\mathbf{b}_0, \mathbf{b}_1)\Big)$$

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We write $C_{TC}(\alpha, \beta; \delta)$ whenever this is true.

- The term condition may be described as a condition that is quantified over a certain invariant relation of $\mathbb{A}$ which is called the algebra of (α, β)-matrices and is denoted $M(\alpha, \beta)$.

Matrices

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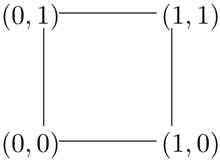

- We say that a relation R on a set A is 2-dimensional if $R \subseteq A^{2^2}$ (R is a set of squares whos vertices are labeled by elements of A.)

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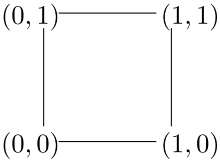

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- $M(\alpha, \beta)$ is the subalgebra of $\mathbb{A}^{2^2}$ with generators

$$\left\{\begin{bmatrix} x & y \\ x & y \end{bmatrix} : x \equiv_\alpha y\right\} \bigcup \left\{\begin{bmatrix} y & y \\ x & x \end{bmatrix} : x \equiv_\beta y\right\}$$

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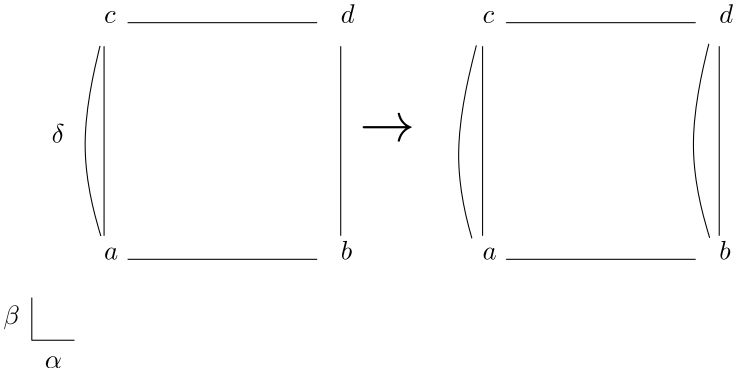
For $\delta \in \text{Con}(\mathbb{A})$ we have that α **centralizes** β **modulo** δ if the implication



holds for all (α, β)-matrices. This condition is abbreviated $C_{TC}(\alpha, \beta; \delta)$.

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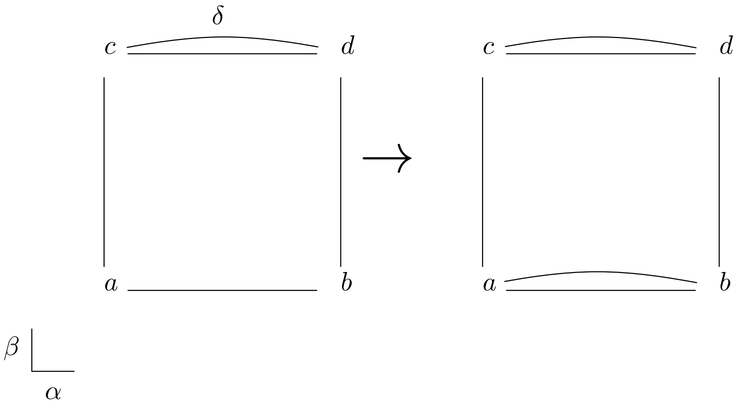
Similarly, we have that β **centralizes** α **modulo** δ if the implication



holds for all (α, β)-matrices. This condition is abbreviated $C_{TC}(\beta, \alpha; \delta)$.

Matrices

- The binary commutator is defined to be

$$[\alpha, \beta]_{TC} = \bigwedge \{\delta : C(\alpha, \beta; \delta)\}$$

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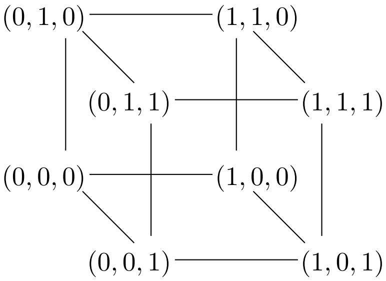

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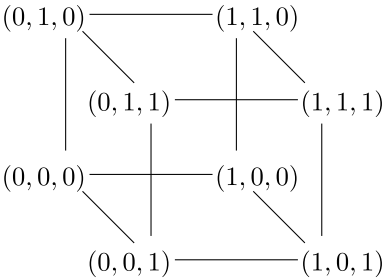
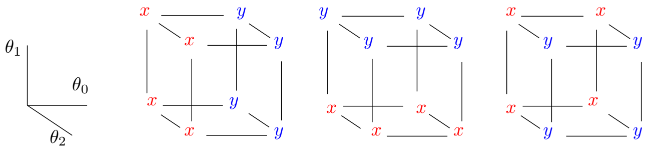


- We say that a relation R on a set A is 3-dimensional if $R \subseteq A^{3^2}$ (R is a set of cubes whos vertices are labeled by elements of A.)

Matrices

- For congruences $\theta_0, \theta_1, \theta_2 \in \mathsf{Con}(\mathbb{A})$, set $M(\theta_0, \theta_1, \theta_2) \leq \mathbb{A}^{2^3}$ to be the subalgebra generated by the following labeled cubes:



$M(\theta_0, \theta_1, \theta_2)$ is called the algebra of $(\theta_0, \theta_1, \theta_2)$-matrices.

Centrality

- For $\delta \in \mathrm{Con}(\mathbb{A})$, we say that θ_0, θ_1 **centralize** θ_2 **modulo** δ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:

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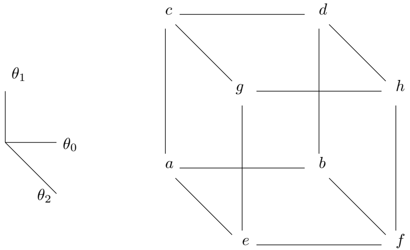

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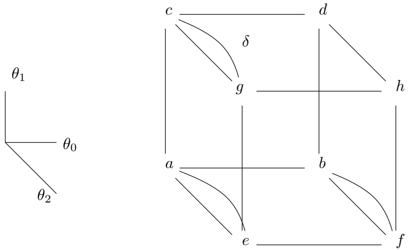

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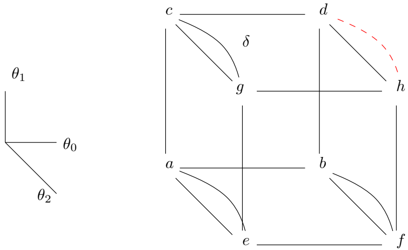

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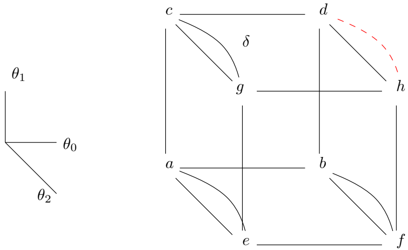
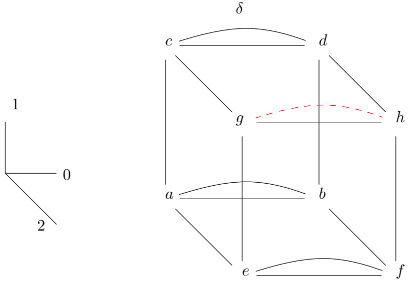


- This condition is abbreviated $C_{TC}(\theta_0, \theta_1, \theta_2; \delta)$.

Centrality

- Here is a picture of $C_{TC}(\theta_1, \theta_2, \theta_0; \delta)$:



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- Higher centrality and the commutator for arity ≥ 4 are similarly defined.

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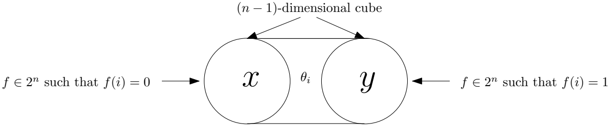
- We say that a relation R on a set A is n-dimensional if $R \subseteq A^{2^n}$
- **Observation:** The term condition definition of centrality involving k-many congruences $\theta_0, \ldots, \theta_{k-1}$ is a condition that is quantified over $(\theta_0, \ldots, \theta_{n-1})$-**matrices**, which are certain n-dimensional invariant relations

$$M(\theta_0, \ldots, \theta_{k-1}) \leq \mathbb{A}^{2^n}$$

that have generators of the form



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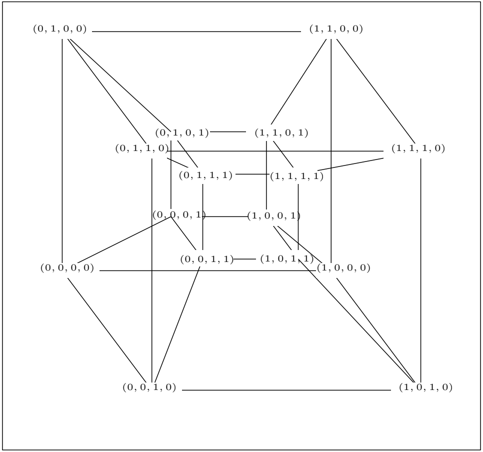

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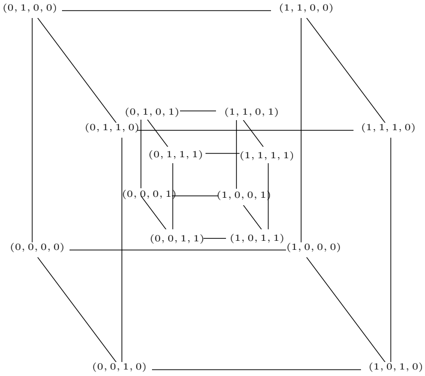

$(\mathbb{H}_n)_3^0$ and $(\mathbb{H}_n)_3^1$

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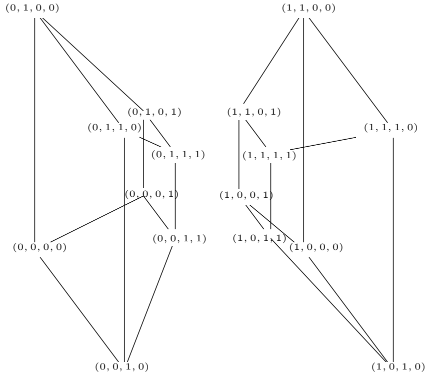

$(\mathbb{H}_n)^0_0$ and $(\mathbb{H}_n)^1_0$

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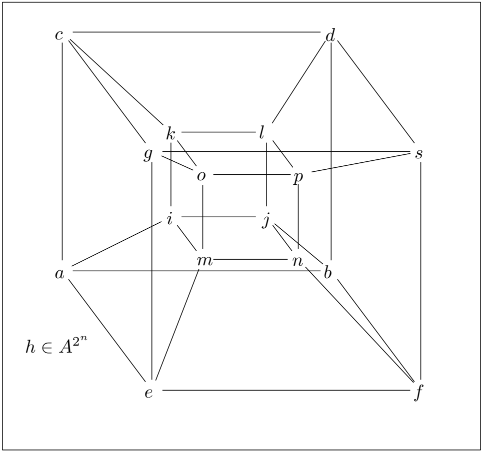

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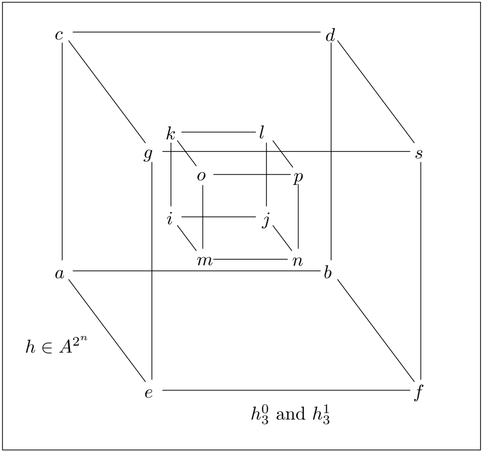

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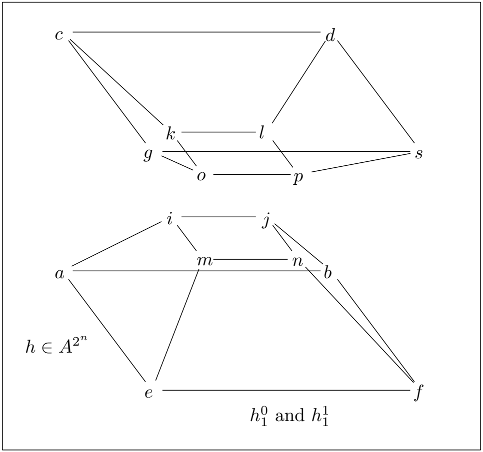

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- **Fact:** Suppose $\mathbb{A}$ is a member of a permutable variety, and take $(\theta_0, \ldots, \theta_{k-1}) \in \mathrm{Con}(\mathbb{A})^n$. Then,

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is a congruence relation, for all $i \in n$.
- This leads to a nice characterization of the commutator for permutable varieties, e.g.,

Theorem (Binary Commutator)

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Let $\mathcal{V}$ *be a permutable variety and let* $\mathbb{A} \in \mathcal{V}$*. For* $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$*, the following are equivalent:*

1. $\langle x, y \rangle \in [\alpha, \beta]_{TC}$

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4. $\begin{bmatrix} x & y \\ b & b \end{bmatrix} \in M(\alpha, \beta)$ *for some* $b \in A$.

- Let $\mathcal{V}$ be a modular variety and let $\mathbb{A} \in \mathcal{V}$. For $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, define $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)_0$.

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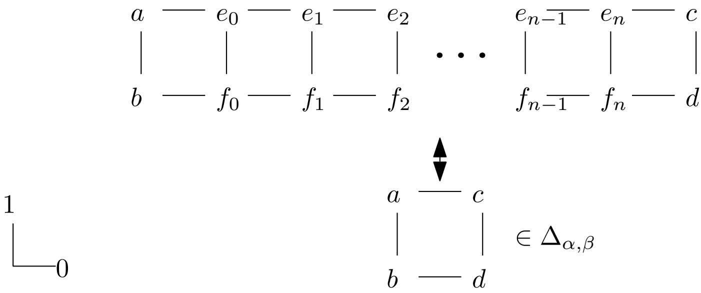

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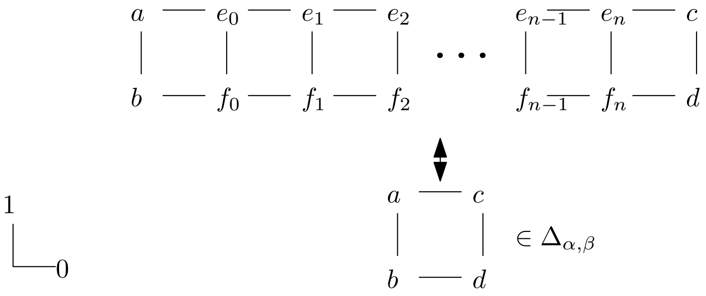

- **Fact:** Both $(\Delta_{\alpha,\beta})_0$ and $(\Delta_{\alpha,\beta})_1$ are congruence relations.

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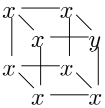
4. $\begin{bmatrix} x & y \\ b & b \end{bmatrix} \in \Delta_{\alpha,\beta}$ *for some* $b \in A$.

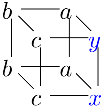
Theorem: Let $\mathcal{V}$ be a permutable variety. Take $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$ for $\mathbb{A} \in \mathcal{V}$. The following are equivalent:

(1) $\langle x, y \rangle \in [\theta_0, \theta_1, \theta_2]$

(2) 

$\in M(\theta_0, \theta_1, \theta_2)$

There exist elements of $\mathbb{A}$ such that

(3) 

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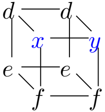
(4) 

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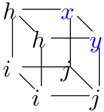
(5) 

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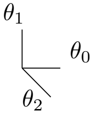
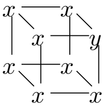


Theorem: Let $\mathcal{V}$ be a modular variety. Take $\theta_0, \theta_1, \theta_2 \in \mathrm{Con}(\mathbb{A})$ for $\mathbb{A} \in \mathcal{V}$. The following are equivalent:

(1) $\langle x, y \rangle \in [\theta_0, \theta_1, \theta_2]$

(2) 

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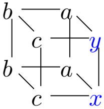
(3) 

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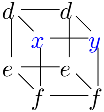
(4) 

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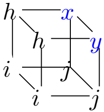
(5) 

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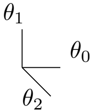

Higher Dimensional Congruence Relations

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Let $R \subseteq A^{2^n}$ be an n-dimensional relation on some set A. R is called an *n*-**dimensional equivalence relation** if for all $i \in n$, each R_i is an equivalence relation.

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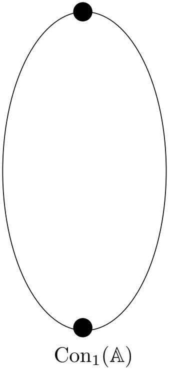
- There are n distinct embeddings from $\mathrm{Con}_1(\mathbb{A})$ into $\mathrm{Con}_n(\mathbb{A})$.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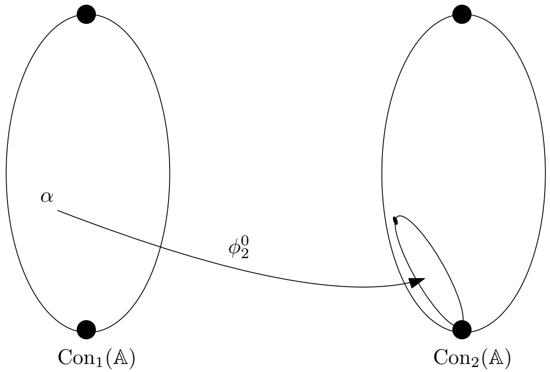
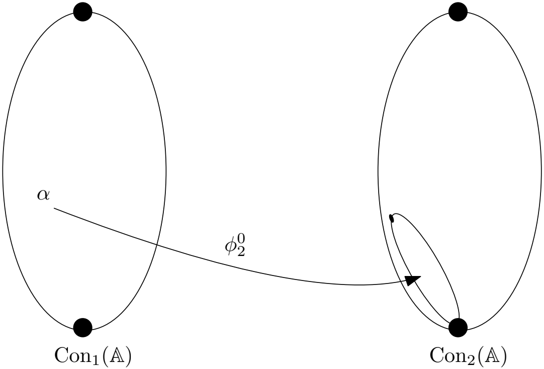
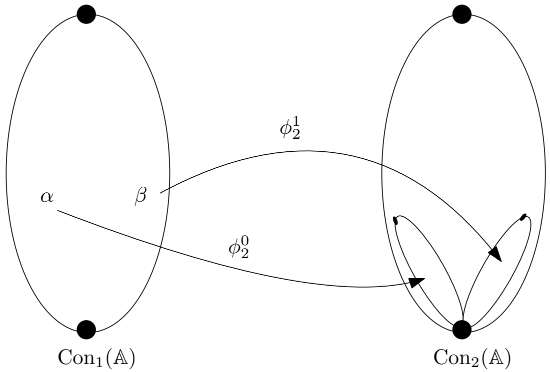


α
ϕ_2^0
$\mathrm{Con}_1(\mathbb{A})$
$\mathrm{Con}_2(\mathbb{A})$



$$\phi_2^0(\alpha) = \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : \langle x, y \rangle \in \alpha \right\}$$



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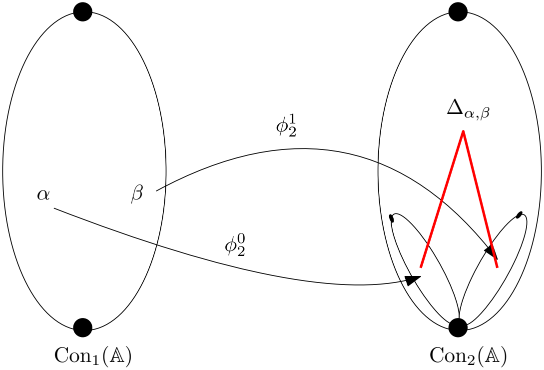
Define $\Delta_{\alpha,\beta} = \phi_2^0(\alpha) \vee \phi_2^1(\beta)$



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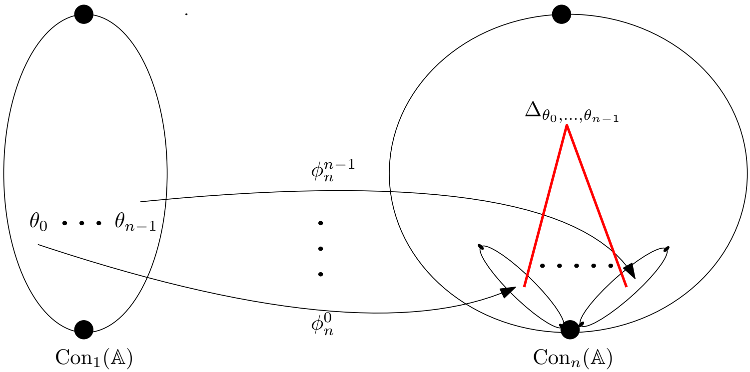
  2. $\big(\mathrm{Cube}_i(\langle x, y \rangle)\big)_i^1$ is the $(n-1)$-dimensional cube with each vertex labeled by y.
- Define $\phi_n^i : \mathrm{Con}_1(\mathbb{A}) \to \mathrm{Con}_n(\mathbb{A})$ by

$$\phi_n^i(\alpha) = \{\mathrm{Cube}_i(\langle x, y \rangle) : \langle x, y \rangle \in \alpha\}$$

Define $\Delta_{\theta_0,\dots,\theta_{n-1}} = \bigvee_i \phi_n^i(\theta_i)$



Characterizing Joins

- Let $\mathbb{A}$ be an algebra and let θ be an equivalence relation on A. Then, θ is an admissible relation if and only if θ is compatible with the unary polynomials of $\mathbb{A}$.

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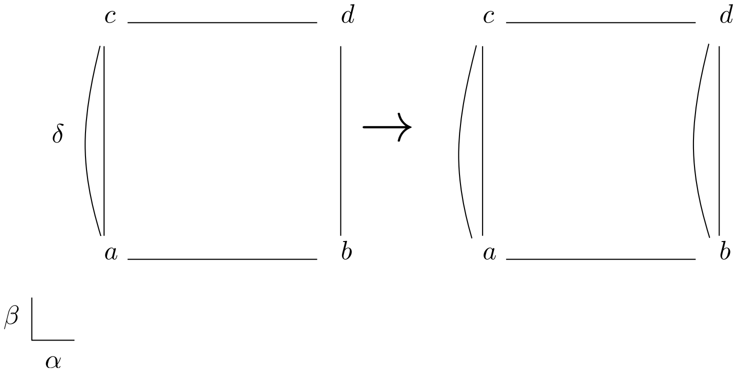
    1. Closing $\bigcup \phi_n^i(\theta_i)$ under all n-ary polynomials and then
    2. taking a sequence of transitive closures, cycling through all possible directions possibly ω-many times.
- Notice: $M(\theta_0, \ldots, \theta_{n-1}) \leq \Delta_{\theta_0,\ldots,\theta_{n-1}}$. We use this larger set to define a stronger term condition.

Hypercentrality

For $\delta \in \mathrm{Con}(\mathbb{A})$ we have that α **hypercentralizes** β **modulo** δ if the implication



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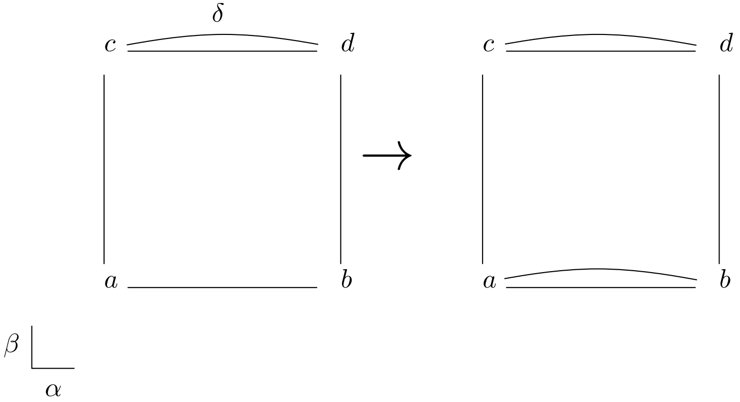

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- Higher arity hypercentrality and the higher arity hypercommutator similarly defined.

Theorem (Binary Hyper Commutator)

Let $\mathbb{A}$ be an algebra. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, the following are equivalent:

1. $\langle x, y \rangle \in [\alpha, \beta]_H$
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- A similar characterization of the higher arity hyper commutator also holds.

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- This is probably true for modular variates (only written up for the ternary case.)

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$$\Delta^L_{\alpha,\beta} = M^*(\alpha, \beta)|_{A^{2^2}}$$

and define $C_L(\alpha, \beta; \delta)$ to be the usual centrality condition quantified over this new set of vertex labeled squares. The linear commutator is now defined in the obvious way.

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- Kearnes and Szendrei showed that $[\alpha, \beta]_S = [\alpha, \beta]_L$ in any Taylor variety. Is this true for higher arity also?
- Can two distinct polynomial clones on a finite set have the same higher dimensional congruences?