Random Models for Idempotent Linear Maltsev Conditions

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$\mathcal{C}_{\mathcal{M}}$ *For each symbol $f \in \mathcal{L}$ there exists a term f (in the language of the given variety or algebra) such that the identities in Σ hold for these terms (in the given variety or algebra).*

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($4^{(4^2)} \cdot 4^4 \approx 10^{12}$ possibilities)

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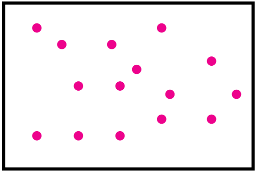

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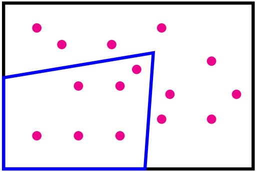

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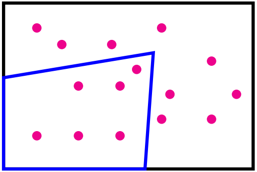

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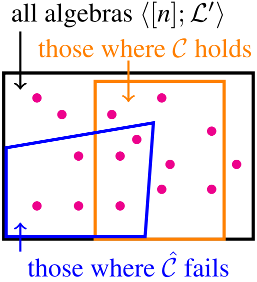

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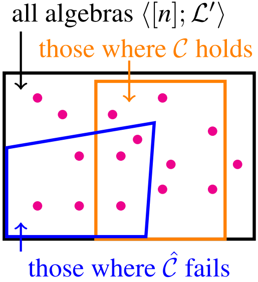

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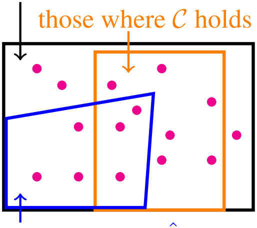

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- On the other hand, if all symbols in $\mathcal{L}'$ have arity 1, then

$$\Pr^{\infty}(\neg\hat{\mathcal{C}} \mid \mathcal{C}) = \begin{cases} 1 & \text{if } \mathcal{C} \text{ is trivial (so } \hat{\mathcal{C}} \text{ is nontrivial)}, \\ \text{undefined} & \text{if } \mathcal{C} \text{ is nontrivial.} \end{cases}$$

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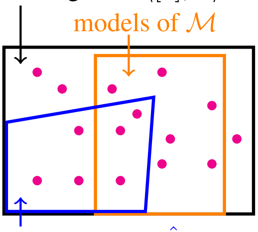

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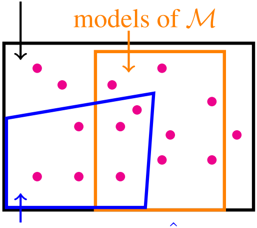

Note: $\Pr^{\infty}(\text{model of } \mathcal{M}) = 0$

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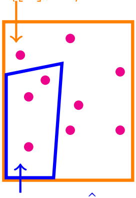

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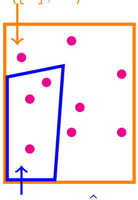

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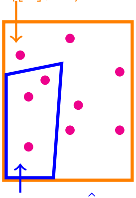

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$X := \{x, y, z\}$ is large enough for $\mathcal{M}$.

Equiv classes of $\overset{\mathcal{M}}{\approx}$, arranged in $\mathrm{Sym}(X)$-orbits:

C_0: $x \overset{\mathcal{M}}{\approx} P_1(x,x,x) \overset{\mathcal{M}}{\approx} P_2(x,x,x) \overset{\mathcal{M}}{\approx} P_1(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,z,z) \overset{\mathcal{M}}{\approx} P_2(y,y,x) \overset{\mathcal{M}}{\approx} P_2(z,z,x)$

—: $y \approx \dots$

—: $z \approx \dots$

X_{C_i}	G_{C_i}

Example: Linear Consequences of Σ

$\mathcal{M} = (\mathcal{L}, \Sigma)$ with $\mathcal{L} := \{P_1, P_2\}$ and

$$\Sigma := \{x \approx P_1(x,y,y),\ P_1(x,x,y) \approx P_2(x,y,y),\ P_2(x,x,y) \approx y,\\ P_1(x,y,z) \approx P_1(x,z,y),\ P_2(x,y,x) \approx P_2(y,x,y)\}.$$

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Equiv classes of $\overset{\mathcal{M}}{\approx}$, arranged in $\mathrm{Sym}(X)$-orbits:

C_0: $x \overset{\mathcal{M}}{\approx} P_1(x,x,x) \overset{\mathcal{M}}{\approx} P_2(x,x,x) \overset{\mathcal{M}}{\approx} P_1(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,z,z) \overset{\mathcal{M}}{\approx} P_2(y,y,x) \overset{\mathcal{M}}{\approx} P_2(z,z,x)$

—: $y \approx \dots$

—: $z \approx \dots$

X_{C_i}	G_{C_i}
$\{x\}$	$\{\mathrm{id}\}$

back

Example: Linear Consequences of Σ

$\mathcal{M} = (\mathcal{L}, \Sigma)$ with $\mathcal{L} := \{P_1, P_2\}$ and

$$\Sigma := \{x \approx P_1(x,y,y),\ P_1(x,x,y) \approx P_2(x,y,y),\ P_2(x,x,y) \approx y,\ P_1(x,y,z) \approx P_1(x,z,y),\ P_2(x,y,x) \approx P_2(y,x,y)\}.$$

$X := \{x, y, z\}$ is large enough for $\mathcal{M}$.

Equiv classes of $\overset{\mathcal{M}}{\approx}$, arranged in $\mathrm{Sym}(X)$-orbits:

C_0: $x \overset{\mathcal{M}}{\approx} P_1(x,x,x) \overset{\mathcal{M}}{\approx} P_2(x,x,x) \overset{\mathcal{M}}{\approx} P_1(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,z,z) \overset{\mathcal{M}}{\approx} P_2(y,y,x) \overset{\mathcal{M}}{\approx} P_2(z,z,x)$

—: $y \approx \ldots$

—: $z \approx \ldots$

C_1: $P_1(x,x,y) \overset{\mathcal{M}}{\approx} P_2(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,y,x)$

X_{C_i}	G_{C_i}
$\{x\}$	$\{\mathrm{id}\}$

back

Example: Linear Consequences of Σ

$\mathcal{M} = (\mathcal{L}, \Sigma)$ with $\mathcal{L} := \{P_1, P_2\}$ and

$$\Sigma := \{x \approx P_1(x,y,y),\ P_1(x,x,y) \approx P_2(x,y,y),\ P_2(x,x,y) \approx y,\ P_1(x,y,z) \approx P_1(x,z,y),\ P_2(x,y,x) \approx P_2(y,x,y)\}.$$

$X := \{x, y, z\}$ is large enough for $\mathcal{M}$.

Equiv classes of $\overset{\mathcal{M}}{\approx}$, arranged in $\mathrm{Sym}(X)$-orbits:

C_0: $x \overset{\mathcal{M}}{\approx} P_1(x,x,x) \overset{\mathcal{M}}{\approx} P_2(x,x,x) \overset{\mathcal{M}}{\approx} P_1(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,z,z) \overset{\mathcal{M}}{\approx} P_2(y,y,x) \overset{\mathcal{M}}{\approx} P_2(z,z,x)$

—: $y \approx \dots$

—: $z \approx \dots$

C_1: $P_1(x,x,y) \overset{\mathcal{M}}{\approx} P_2(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,y,x)$

—: $P_1(y,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,x) \overset{\mathcal{M}}{\approx} P_1(y,x,y)$

⋮

—: $P_1(z,z,y) \overset{\mathcal{M}}{\approx} \dots$

X_{C_i}	G_{C_i}
$\{x\}$	$\{\mathrm{id}\}$

back

Example: Linear Consequences of Σ

$\mathcal{M} = (\mathcal{L}, \Sigma)$ with $\mathcal{L} := \{P_1, P_2\}$ and
$\Sigma := \{x \approx P_1(x,y,y),\ P_1(x,x,y) \approx P_2(x,y,y),\ P_2(x,x,y) \approx y,$
$P_1(x,y,z) \approx P_1(x,z,y),\ P_2(x,y,x) \approx P_2(y,x,y)\}.$

$X := \{x, y, z\}$ is large enough for $\mathcal{M}$.

Equiv classes of $\overset{\mathcal{M}}{\approx}$, arranged in $\mathrm{Sym}(X)$-orbits:

		X_{C_i}	G_{C_i}
C_0:	$x \overset{\mathcal{M}}{\approx} P_1(x,x,x) \overset{\mathcal{M}}{\approx} P_2(x,x,x) \overset{\mathcal{M}}{\approx} P_1(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,z,z) \overset{\mathcal{M}}{\approx} P_2(y,y,x) \overset{\mathcal{M}}{\approx} P_2(z,z,x)$	$\{x\}$	$\{\mathrm{id}\}$
—:	$y \approx \ldots$		
—:	$z \approx \ldots$		
C_1:	$P_1(x,x,y) \overset{\mathcal{M}}{\approx} P_2(x,y,y) \overset{\mathcal{M}}{\approx} P_1(x,y,x)$	$\{x, y\}$	$\{\mathrm{id}\}$
—:	$P_1(y,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,x) \overset{\mathcal{M}}{\approx} P_1(y,x,y)$		
$\vdots$			
—:	$P_1(z,z,y) \overset{\mathcal{M}}{\approx} \ldots$		

back

back

Example: Linear Consequences of Σ (cont'd)

C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$

Example: Linear Consequences of Σ (cont'd)

C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$

—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$

—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$

back

Example: Linear Consequences of Σ (cont'd)

C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$

—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$

—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$

◂ back

C_3: $P_1(x,y,z) \overset{\mathcal{M}}{\approx} P_1(x,z,y)$

Example: Linear Consequences of Σ (cont'd)

C_2: $P_2(x, y, x) \overset{\mathcal{M}}{\approx} P_2(y, x, y)$

—: $P_2(x, z, x) \overset{\mathcal{M}}{\approx} P_2(z, x, z)$

—: $P_2(y, z, y) \overset{\mathcal{M}}{\approx} P_2(z, y, z)$

C_3: $P_1(x, y, z) \overset{\mathcal{M}}{\approx} P_1(x, z, y)$

—: $P_1(y, x, z) \overset{\mathcal{M}}{\approx} P_1(y, z, x)$

—: $P_1(z, x, y) \overset{\mathcal{M}}{\approx} P_1(z, y, x)$

back

Example: Linear Consequences of Σ (cont'd)

C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$

—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$

—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$

back

C_3: $P_1(x,y,z) \overset{\mathcal{M}}{\approx} P_1(x,z,y)$

—: $P_1(y,x,z) \overset{\mathcal{M}}{\approx} P_1(y,z,x)$

—: $P_1(z,x,y) \overset{\mathcal{M}}{\approx} P_1(z,y,x)$

C_4: $P_2(x,y,z)$

Example: Linear Consequences of Σ (cont'd)

X_{C_i} G_{C_i}

C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$

—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$

—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$

back

C_3: $P_1(x,y,z) \overset{\mathcal{M}}{\approx} P_1(x,z,y)$

—: $P_1(y,x,z) \overset{\mathcal{M}}{\approx} P_1(y,z,x)$

—: $P_1(z,x,y) \overset{\mathcal{M}}{\approx} P_1(z,y,x)$

C_4: $P_2(x,y,z)$

—: $P_2(x,z,y)$

⋮

—: $P_2(z,y,x)$

back

Example: Linear Consequences of Σ (cont'd)

C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$

—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$

—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$

C_3: $P_1(x,y,z) \overset{\mathcal{M}}{\approx} P_1(x,z,y)$

—: $P_1(y,x,z) \overset{\mathcal{M}}{\approx} P_1(y,z,x)$

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C_4: $P_2(x,y,z)$

—: $P_2(x,z,y)$

⋮

—: $P_2(z,y,x)$

X_{C_i}	G_{C_i}
$\{x,y\}$	$\{\mathrm{id},\ (x\ y)\}$

back

back

Example: Linear Consequences of Σ (cont'd)

	X_{C_i}	G_{C_i}
C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$	$\{x,y\}$	$\{\mathrm{id},\ (x\ y)\}$
—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$		
—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$		back
C_3: $P_1(x,y,z) \overset{\mathcal{M}}{\approx} P_1(x,z,y)$	$\{x,y,z\}$	$\{\mathrm{id},\ (y\ z)\}$
—: $P_1(y,x,z) \overset{\mathcal{M}}{\approx} P_1(y,z,x)$		
—: $P_1(z,x,y) \overset{\mathcal{M}}{\approx} P_1(z,y,x)$		
C_4: $P_2(x,y,z)$		
—: $P_2(x,z,y)$		
⋮		
—: $P_2(z,y,x)$		back

Example: Linear Consequences of Σ (cont'd)

	X_{C_i}	G_{C_i}
C_2: $P_2(x,y,x) \overset{\mathcal{M}}{\approx} P_2(y,x,y)$	$\{x,y\}$	$\{\mathrm{id},\ (x\ y)\}$
—: $P_2(x,z,x) \overset{\mathcal{M}}{\approx} P_2(z,x,z)$		
—: $P_2(y,z,y) \overset{\mathcal{M}}{\approx} P_2(z,y,z)$		back
C_3: $P_1(x,y,z) \overset{\mathcal{M}}{\approx} P_1(x,z,y)$	$\{x,y,z\}$	$\{\mathrm{id},\ (y\ z)\}$
—: $P_1(y,x,z) \overset{\mathcal{M}}{\approx} P_1(y,z,x)$		
—: $P_1(z,x,y) \overset{\mathcal{M}}{\approx} P_1(z,y,x)$		
C_4: $P_2(x,y,z)$	$\{x,y,z\}$	$\{\mathrm{id}\}$
—: $P_2(x,z,y)$		
$\vdots$		
—: $P_2(z,y,x)$		back

Example: Constructing Random Models of $\mathcal{M}$

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$$\mathcal{M} = (\mathcal{L}, \Sigma),\ \Sigma := \{x \approx P_1(x,y,y),\ P_1(x,x,y) \approx P_2(x,y,y),\ P_2(x,x,y) \approx y,\ P_1(x,y,z) \approx P_1(x,z,y),\ P_2(x,y,x) \approx P_2(y,x,y)\}$$

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$$A \xleftarrow{\quad P_1 \quad} A^3 \qquad\qquad A^3 \xrightarrow{\quad P_2 \quad} A$$

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$\mathbf{A} = \langle A; P_1, P_2 \rangle$ is a model of $\mathcal{M}$ iff P_1, P_2 have the foll. form $(a, b, c$ distinct)

$$A \xleftarrow{\ P_1\ } A^3 \qquad\qquad A^3 \xrightarrow{\ P_2\ } A$$

aaa	*aaa*
abb	*bba*
abc	*abc*
aba	*aba*
aab	*abb*

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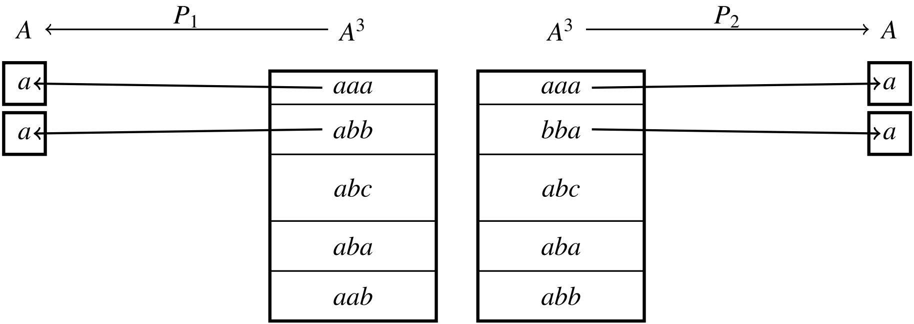

check

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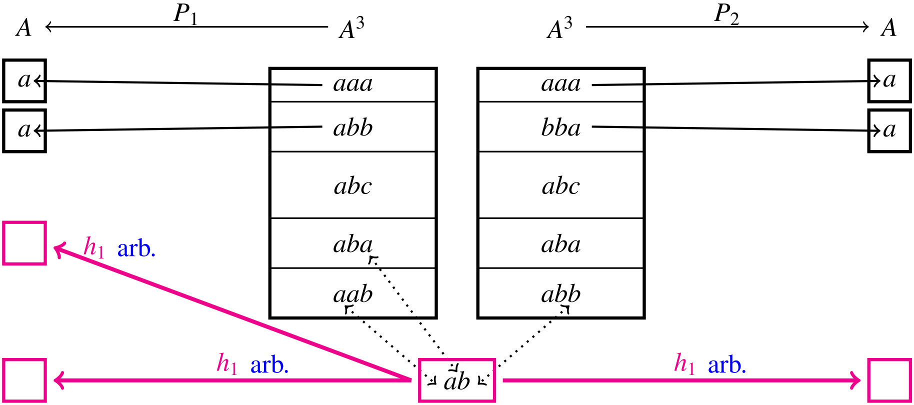

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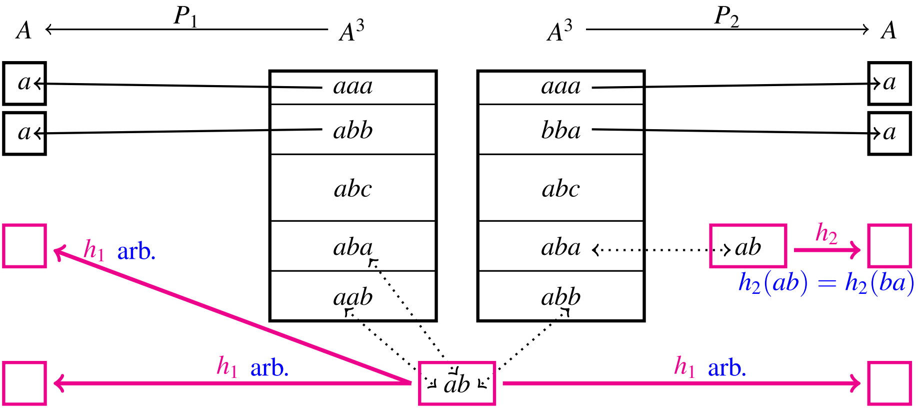

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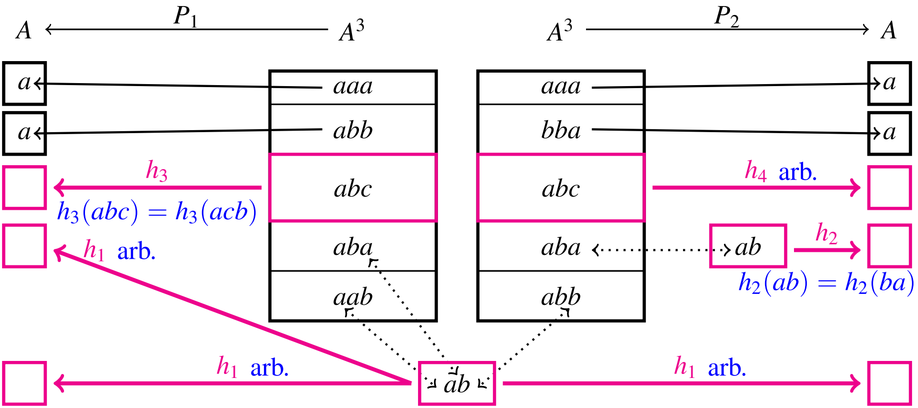

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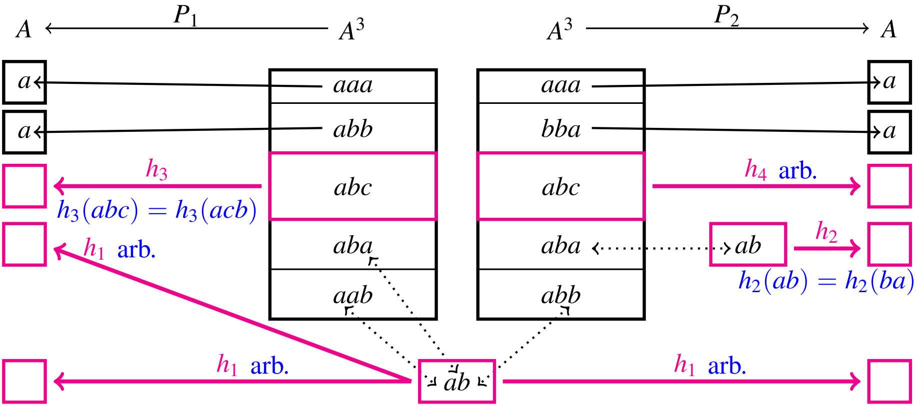

Also: h_1, h_2, h_3, h_4 are independent.

check check

Constructing Random Models of $\mathcal{M}$

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Let $\mathcal{M} = (\mathcal{L}, \Sigma)$ be as before the ex's, and $X = \{x_1, \ldots, x_m\}$ large enough.

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Assume also (WLOG) that t_i depends on all d_i variables (mod Σ), and

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Theorem

For any set A, the map $\mathbf{A} \mapsto (t_i{\restriction}A^{(d_i)})_{1 \leq i \leq r}$ is a bijection between the models of $\mathcal{M}$ on A and the r-tuples $(h_i)_{1 \leq i \leq r}$ of functions $h_i \colon A^{(d_i)} \to A$ such that h_i is invariant under all permutations $\pi \in G_{[t_i]}$ of its variables.

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- *The functions in each such r-tuple $(h_i)_{1\leq i\leq r}$ are independent;*

Constructing Random Models of $\mathcal{M}$

Let $\mathcal{M} = (\mathcal{L}, \Sigma)$ be as before the ex's, and $X = \{x_1, \ldots, x_m\}$ large enough.
For any set A and $k \geq 1$, let $A^{(k)} := \{(a_1, \ldots, a_k) \in A^k : a_1, \ldots, a_k \text{ distinct}\}$.

Fix $t_i = t_i(x_1, \ldots, x_{d_i})$ $(1 \leq i \leq r)$ so that they form a maximal family of *essentially different, nontrivial* linear $\mathcal{L}$-terms, i.e.,

- $[t_1], \ldots, [t_r], [x_1]$ is a transversal for the $\mathrm{Sym}(X)$-orbits of the $\overset{\mathcal{M}}{\approx}_X$-blocks.

Assume also (WLOG) that t_i depends on all d_i variables (mod Σ), and

$$(2 \leq)\, d := d_1 = \cdots = d_\ell < d_{\ell+1} \leq \cdots \leq d_r.$$

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- *The functions in each such r-tuple $(h_i)_{1\leq i\leq r}$ are independent; moreover*
- *for every i, we have $h_i = \bigcup\{h_i{\restriction}D^{(d_i)} : D \in \binom{A}{d_i}\}$ where the functions $h_i{\restriction}D^{(d_i)}$ $(D \in \binom{A}{d_i})$ are independent.*

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  - In particular, if $\mathcal{M}$ is the system for congr 3-permutability, then the probability that a random finite model of $\mathcal{M}$ has no Maltsev term is 0. This answers first question in Slide 1

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For $k \geq d$ let $\quad p_{\mathcal{M}}(k) := \sum_{i=1}^{r} q_i \binom{k}{d_i} \quad$ where $q_i = |\mathrm{Sym}(x_1, \ldots, x_{d_i}) : G_{[t_i]}|$.

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back back

Idea of Proof of the Theorem on Subalgebras

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- Similar, using that X_u compatible implies:
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- Hence, the probability that a random finite algebra $\langle A; P\rangle$ with a Maltsev operation P fails to have a majority term is $1 - 1/\sqrt{e}$.
  This answers second question in Slide 1

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Problem

Find all random finite models of $\mathcal{M}$ (up to term equivalence) which occur with positive probability.