Chapter 1

A Context for Calculus

"Applications are not just something to stick at the end of the chapter."

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1.1 Introduction: What is Calculus?

Calculus was invented (discovered?) by Isaac Newton and Gottfried Leibniz, more or less independently, in the mid-to-late-seventeenth century. It is often described as the mathematics of *change.* But what does this mean? How does the mathematics of change **work**; what are the **central principles** behind the mathematics of change? And why is it **important** to have a mathematics of change?

Approaches to these questions will constitute the bulk of this book. And even then, we'll really only scratch the surface. In this section, we'll attempt to scratch the surface of that surface.

We begin with the latter question, concerning the importance of calculus. To address this question, we make the rather transparent observation that things are always changing. And in "real life" – or in science, anyway, which is both easier and harder, both cleaner and messier, than real life – we often hope to *predict* change.

We may seek to predict progression of a weather phenomenon, or growth of a population, or decay of a radioactive substance, or activation of a protein, or trajectory of a rocket, or interaction of chemicals, or diffusion of a substance across a membrane, or evolution of an epidemic, or neural response to an electrical impulse. And so on.

If we can predict the behavior of a phenomenon, then we can prepare for, alter, divert, or even embrace that phenomenon. If we can predict which kind of outcome will result from which type of initial scenario, then – assuming we have influence over the starting conditions – we can implement the initial scenario that will yield the most favorable outcome.

Certainly, not all of science (or real life) is about prediction. We often wish to study the past, or the present, as much of the future. Yet investigations of the past, and examinations of the present, are themselves often performed with an eye towards knowing and impacting the future. Moreover, analyses of the past – for example, approximating the age of an artifact using carbon dating – are often achieved through the application of predictive models that work just as well in either time direction.

Which brings us to the above questions about the workings and principles of calculus. We take the point of view, in this book, that these all hinge in some way on the following fundamental idea, which is, at its heart, all about prediction.

Declaration. Calculus *is* the set of mathematical manifestations, implications, and extensions of the fact that:

> *If you know how fast you're going, then you know how far you'll get in a given amount of time.*

To clarify this declaration, we wish to develop it into to an equation – the **prediction equation**, which will take a variety of forms, and which will, in these various forms, be central to this book.

To this end, imagine an object moving with speed s. Denote the distance traveled by this object by y, and time by the variable t. Also, as is common in mathematics, denote a *change* in a quantity by putting a "Δ" ("delta") in front of that quantity. Then the above declaration may be encapsulated by:

$$\boxed{\boldsymbol{\Delta y = s\,\Delta t}}$$

The prediction equation, preliminary version (PEPV)

That is, "change in distance equals speed times elapsed time."

Example 1.1.1. A car moving with speed $s = 60$ miles (mi) per hour (hr), for an interval of $\Delta t = 2.5$ hr, travels a distance of

$$\Delta y = s\,\Delta t = 60\ \frac{\text{mi}}{\text{hr}} \times 2.5 \text{ hr} = 60 \times 2.5 \text{ mi} = 150 \text{ mi}.$$

Note how the units cancel: mi/hr times hr equals mi. ("Per" always means "divided by.") "Unit analysis" provides a good check on our choice of mathematical processes. That is: the fact that the units work out properly, so that the final units make sense to the problem, is evidence that we're at least doing some things right.

Upon reading Example 1.1.1, do you think to yourself "But what happens if s is changing?," or something to that effect? This question is *absolutely central* to calculus – as central as is the above prediction equation itself.

As a first attempt to answer, let's imagine that we're driving down the road and, at some particular point in our journey, we observe a speedometer reading of $s = 60$ mph (mi/hr). It's unlikely that, over the next 2.5 hours, we will travel exactly 150 miles, because again, our speed is itself changing. On other hand, except under extreme circumstances, our speed is not likely to change too much over *short* intervals of time. So there's a good chance that, over such intervals, the above prediction equation will give us good *approximations* to distance traveled.

This leads us to the following revision of our prediction equation:

$$\boxed{\boldsymbol{\Delta y \approx s\,\Delta t}}$$

The prediction equation, approximate version (PEAV)

The "$\approx$" symbol here is read "is approximately equal to." Implicit in this (approximate) equation is the idea that, the smaller we take Δt, the better an approximation we get to Δy. (By "better" we mean: the ratio of the error in the approximation to the actual distance traveled should be smaller.)

Example 1.1.2. At a particular instant, a car's speedometer shows a speed of $s = 60$ mph. Assuming no excessively rapid changes in the car's speed, about how far does the car travel over the next ten seconds? Answer (a) in miles, and (b) in feet (ft).

Solution. (a) We first note that one hour equals 3600 seconds (sec), so 1 sec=1/3600 hr. So 10 sec= 10/3600 =1/360 hr. So, in miles,

$$\Delta y = s\,\Delta t = 60\ \frac{\text{mi}}{\text{hr}} \times 10 \text{ sec} = 60\ \frac{\text{mi}}{\text{hr}} \times \frac{1}{360}\text{ hr} = \frac{60}{360}\text{ mi} = \frac{1}{6}\text{ mi}.$$

(b) We have 1 mi=5280 ft, so, by part (a),

$$\Delta y = \frac{1}{6}\text{ mi} = \frac{1}{6}\text{ mi} \times 5280\ \frac{\text{ft}}{\text{mi}} = \frac{5280}{6}\text{ ft} = 880 \text{ ft}.$$

(Comparing parts (a) and (b) amounts the sometimes-useful conversion formula 60 mph = 88 ft/sec.)

We would expect that our above equation (PEAV) would give us a *better* estimate of distance traveled over a five-second interval, and an even better one over a one-second interval, and so on.

At this point we might ask: if smaller intervals yield better approximations, then wouldn't the *smallest* possible interval – meaning $\Delta t = 0$ – yield the *best* possible approximation? Well, yes and no. *No* because, if no time elapses, then nothing happens, so putting $\Delta t = 0$ into our approximate equation (PEAV) gives

$$0 \approx s \times 0,$$

which is not interesting or useful. But at the same time *yes* because, if we first *divide* both sides of (PEAV) by Δt, then we obtain the approximate equality

$$\boxed{\boldsymbol{s \approx \frac{\Delta y}{\Delta t}}}$$

The prediction equation, quotient version (PEQV)

And sense *can* be made of (PEQV) when $\Delta t = 0$, or more formally, "in the limit as Δt *approaches* zero," the precise mathematical meaning of which will be discussed in Chapter 2.

Ascribing such a precise meaning to (PEQV), and to its generalizations (see equation (PEGQV) below), as Δt approaches zero, is often considered one of the crowning achievements of calculus. Indeed, many consider this achievement to represent a pinnacle of mathematical, and even metaphysical, thought in general.

Some might argue that calculus is *fundamentally* about limits as Δt approaches zero, and *much less* about what happens for particular non-zero real numbers Δt "along the way." Some might, but we wouldn't. Situations where Δt approaches zero and situations where it doesn't will both occupy major portions of this book. In that spirit, let's return for now to (PEAV), and to the situation where Δt is, again, a non-zero real number.

One major difficulty here is that, as discussed above, we can't expect (PEAV) to provide good results over *large* intervals. There is a solution to this difficulty, at least in situations where we can repeatedly recalibrate, or reassess, our speed s. Namely, in such situations, we can apply this prediction equation *iteratively*. The following example illustrates this iterative process.

Example 1.1.3. Speedometer readings are recorded at ten-second intervals during a car's minute-long trip. Suppose the readings are as follows:

time t (sec)	0	10	20	20	40	50
speed s (mph)	70	65	70	63	68	69

Estimate the total distance traveled, in miles, over this minute.

Solution. We use the speedometer reading at the beginning of each 10-second interval, together with our above, approximate prediction equation (PEAV), to estimate distance traveled over each such interval. We then add up our results to estimate the total distance Δy traveled. We get

$$\begin{aligned}\Delta y \approx{}& 70\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec} + 65\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec} + 70\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec}\\ &+ 63\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec} + 68\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec} + 69\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec}\end{aligned}$$

or, doing some algebra and recalling that 10 sec$=1/360$ hr,

$$\begin{aligned}\Delta y &\approx (70 + 65 + 70 + 63 + 68 + 69)\ \frac{\text{mi}}{\text{hr}} \times 10\ \text{sec}\\ &= 405\ \frac{\text{mi}}{\text{hr}} \times \frac{1}{360}\ \text{hr} = \frac{405}{360}\ \text{mi} = 1.125\ \text{mi}.\end{aligned}$$

The above example may bring to mind a number of questions.

(a) Why don't these people just use their odometer?

(b) If we have to check our speedometer every ten seconds anyway, then how "predictive" is the above algorithm, really?

(c) These computations are pretty tedious, aren't they? And imagine applying the above procedure to a longer journey, of total time several hours, and/or with a smaller "stepsize" Δt, say equal to some fraction of a second, used at each iteration. Yikes!

Regarding question (c), the answer is *yes*, they are tedious – though much less tedious in present times, with the technology available to us, than similar computations were back in the day, when they had to be done "by hand."

We'll see later that calculus gives us, in many cases, a way of *precisely* determining distance information from speed information, and doing so "in one fell swoop" – that is, without having to resort to iteration. In a similar way calculus will allow us to predict, without requiring approximate, iterative procedures, the behavior of all sorts of phenomena using information about the "speeds," or *rates*, at which those phenomena are changing. Indeed calculus was invented (discovered?), in large part, expressly for the purpose of this kind of prediction. And central to accomplishing this purpose is the idea of a *limit* mentioned above.

Again, we'll return to these ideas later, but for now, back to our road trip, and to Example 1.1.3 and questions (a) and (b), above. Certainly, in "real life," we'd expect our car to have a working odometer. (It's the law!) And in this case, we could calculate distance traveled by subtracting the odometer reading at the beginning of the journey from that at the end, and be done with it. This, of course, would completely obviate the need for such an iterative process as was employed in Example 1.1.3 above.

But in real life, or in its close cousin science, one often *does not* have an odometer – or even a speedometer, for that matter. Not literally, at least. But fortunately, one *does* often have, in real life or in science, something that works similarly. Namely, one often has a *rate equation*. What this means, mathematically speaking, is that one often has a formula describing one's speed.

We'll consider a (more or less) "real life" situation involving a rate equation just a bit later in this section. But first, to highlight the main ideas, let's see how rate equations might work in a familiar setting. Specifically, let's return to our car – but, in temporary defiance of real-life requirements, let's replace its speedometer and odometer by a rate equation.

Example 1.1.4. Consider a car traveling for a period of one minute. Let's agree that we are measuring time in seconds and distance in feet. Now suppose we know that the car's speed s, at any point in time during this minute, equals 0.01 times the distance y traveled up to that point in time, plus 40.

Using an iterative procedure with 15-second intervals, estimate the distance traveled by the car over this minute.

Solution. Let's write $y(t)$ and $s(t)$, respectively, for the distance y and speed s at a particular point in time t, within our minute-long interval. Then the given information implies the rate equation

$$s(t) = 0.01y(t) + 40, \tag{1.1.1}$$

where $s(t)$ is in ft/sec and $y(t)$ is in ft.

At $t = 0$ seconds, the distance traveled so far is $y(0) = 0$. So by (1.1.1),

$$s(0) = 0.01\, y(0) + 40 = 0.01 \times 0 + 40 = 40 \text{ ft/sec}.$$

Using this information we *estimate*, using the prediction equation (PEAV), that

$$\Delta y \approx s(0)\, \Delta t = 40\, \frac{\text{ft}}{\text{sec}} \times 15 \text{ sec} = 600 \text{ ft},$$

over the first 15 seconds.

Now what about the next 15 seconds? We don't know *exactly* how far our car has traveled after 15 seconds, so we can't use (1.1.1) to determine *exactly* the speed at $t = 15$. So we do the next best thing: we input, into (1.1.1), our *approximate* distance traveled so far, namely $y(15) \approx 600$ ft!

That is: by (1.1.1),

$$v(15) \approx y(15)\, \Delta t \approx 0.01 \times 600 + 40 = 46 \text{ ft/sec}.$$

We input this (approximate) value of s into (PEAV), to find that

$$\Delta y \approx s\, \Delta t \approx 46\, \frac{\text{ft}}{\text{sec}} \times 15 \text{ sec} = 690 \text{ ft},$$

over the *next* 15 seconds.

Keep in mind that our rate equation (1.1.1) takes *total distance y traveled so far* as input. At time $t = 30$ sec, this total distance y equals, approximately, the sum of the distances traveled over each of the first two 15-second intervals. That is $y(30) \approx 600 + 690 = 1{,}290$ ft. Consequently, by (1.1.1), our speed s at $t = 30$ is given by

$$s(30) \approx 0.01\, y(30) + 40 \approx 0.01 \times 1{,}290 + 40 = 52.9 \text{ ft/sec}.$$

So again by (PEAV), we have

$$\Delta y \approx s\, \Delta t \approx 52.9\, \frac{\text{ft}}{\text{sec}} \times 15 \text{ sec} = 793.5 \text{ ft},$$

over the *third* 15-second interval.

Finally: our total distance traveled, by the end of the first 45 seconds, is $y(45) \approx 1{,}290 + 793.5 = 2{,}083.5$ ft, so by (1.1.1),

$$s(45) \approx 0.01\, y(45) + 40 \approx 0.01 \times 2{,}083.5 + 40 = 60.835 \text{ ft/sec},$$

so that, by (PEAV), our distance traveled over the last 15 fifteen seconds is

$$\Delta y \approx s(45)\, \Delta t \approx 60.835\, \frac{\text{ft}}{\text{sec}} \times 15 \text{ sec} = 912.525 \text{ ft}.$$

Then our total distance traveled over the *entire* one-minute trip is approximately

$$y(60) = 2{,}083.5 + 912.525 = 2{,}996.025 \text{ ft}$$

(or 2,996.025/5,280=0.5674 mi, to four decimal places).

As a "gut check" on the above calculations, we note that, at the outset, our car was traveling at 40 ft/sec. So, were its speed to hold constant, it would travel $40 \times 60 = 2{,}400$ feet in one minute. But its speed is in fact *increasing*; our rate equation (1.1.1) tells us that s gets larger as y does. So it's not surprising that, over the course of the minute, the car travels somewhat more than 2,400 ft. (But not too much more, since the factor of 0.01 in the equation $s(t) = 0.01y(t) + 40$ tells us that s increases relatively slowly with y.)

To conclude this section, and as a prelude to the next one, we wish to switch gears (pun intended), and to investigate a somewhat more realistic scenario. But to do so, we will first require an appropriate, more general interpretation of (PEAV).

For this, we note that speed s may be thought of as *the rate of change of distance y with respect to time t.* So (PEAV) says:

> *change in distance is approximately equal to the* **rate of change** *of distance with respect to time, times elapsed time.*

Now given *any* quantity y varying with time, it is common to denote the rate of change of y, with respect to time, by y'. Then (PEAV) reads

$$\boxed{\boldsymbol{\Delta y \approx y' \, \Delta t}}$$

The prediction equation, general version (PEGV)

Again, the meaning of this is that

> *change in a quantity y is approximately equal to the* **rate of change** *of y with respect to time, times elapsed time.*

But now, we allow y to be essentially anything (that can be measured in real numbers), not necessarily just distance traveled.

We can also divide (PEGV) through by Δt, to obtain the following generalization of (PEQV):

$$\boxed{\boldsymbol{y' \approx \frac{\Delta y}{\Delta t}}}$$

The prediction equation, general quotient version (PEGQV)

As before, (PEGV) are (PEGQV) are *approximate* equalities because the rate of change y' is, itself, typically changing. And all of the other discussions, calculations, and observations above now carry over to our more general context.

To illustrate this, we apply (PEGV) to the following example.

Example 1.1.5. Consider a contagious illness that has spread through some population, by a virus for example. Suppose that, on average, it takes twenty days to recover from this illness. Also suppose, for simplicity, that the numbers of infected and recovered individuals can change only through the process of recovery – everyone who might become ill already has; once recovered, one is immune and can not infect others; no one dies from the illness (or otherwise); and so on.

At a certain point in time, call it $t = 0$, there are 54,000 individuals still infected with the disease, and 6,000 who have recovered. Using all of this information:

(a) Set up a rate equation for the population R of recovered individuals. More specificially, express R', the rate of change of R, in terms of R.

(b) Estimate how many will be recovered after one day, using a "stepsize" Δt equal to one third of a day.

Solution. (a) It's easier to first relate R' to the number *infected* individuals, as follows. Since the illness lasts an average of twenty days, we can conclude that, on any given day, about *one twentieth* of the infected population recovers. That is, the rate of recovery R', in individuals per day, is given (approximately) by

$$R' = \frac{1}{20} I, \tag{1.1.2}$$

where I is the number of infected individuals.

Now remember that $R(0) = 6{,}000$ and $I(0) = 54{,}000$, so that $R(0) + I(0) = 60{,}000$. And we've assumed that only recovery can affect I and R, which means $R + I$ must remain equal to 60,000 at all times. That is, at any point in time, we have $I = 60{,}000 - R$, so that, by equation (1.1.2),

$$R' = \frac{1}{20}(60{,}000 - R). \tag{1.1.3}$$

(b) We're given that $R(0) = 6{,}000$, so by (1.1.3),

$$R'(0) = \frac{1}{20}(60{,}000 - R(0)) = \frac{1}{20}(60{,}000 - 6{,}000) = \frac{54{,}000}{20} = 2{,}700 \text{ individuals/day.}$$

So by (PEGV), over the first third of a day,

$$\Delta R \approx R'(0)\,\Delta t = 2{,}700 \times \frac{1}{3} = 900 \text{ individuals.}$$

Since we started out with $R(0) = 6{,}000$, we can compute, approximately, the number of recovered individuals after a third of a day:

$$R(1/3) \approx R(0) + \Delta R = 6{,}000 + 900 = 6{,}900 \text{ individuals.}$$

We now repeat this process, but with our "new" (approximate) value $R(1/3) \approx 6{,}900$. By (1.1.3),

$$R'(1/3) = \frac{1}{20}(60{,}000 - R(1/3)) \approx \frac{1}{20}(60{,}000 - 6{,}900) = \frac{53{,}100}{20} = 2{,}655 \text{ individuals/day.}$$

So by (PEGV), over the second third of a day,

$$\Delta R \approx R'(1/3)\,\Delta t \approx 2{,}665 \times \frac{1}{3} = 885 \text{ individuals;}$$

so

$$R(2/3) \approx R(1/3) + \Delta R = 6{,}900 + 885 = 7{,}785 \text{ individuals.}$$

One more time: by (1.1.3),

$$R'(2/3) = \frac{1}{20}(60{,}000 - R(2/3)) \approx \frac{1}{20}(60{,}000 - 7{,}785) = \frac{52{,}215}{20} = 2{,}610.75 \text{ individuals/day.}$$

So by (PEGV), over the last third of a day,

$$\Delta R \approx R'(2/3)\,\Delta t \approx 2{,}610.75 \times \frac{1}{3} = 870.25 \text{ individuals,}$$

so

$$R(1) \approx R(2/3) + \Delta R = 7{,}785 + 870.25 = 8{,}655.25 \text{ individuals.}$$

In sum: by the end of the first day, there are about 8,655.25 recovered individuals.

We make a number of observations concerning the above example.

1. A quicker estimate of $R(1)$ can be obtained by arguing as follows. If we begin with 54,000 infected and 6,000 recovered, and the infected population decreases by a factor of $1/20$ per day, then we would expect to have

$$6{,}000 + \frac{1}{20} \times 54{,}000 = 8{,}700$$

   recovered after one day. But this argument, which employs stepsize $\Delta t = 1$, does not take into account that R and I (and therefore R') are themselves changing over the course of a day.

   In our above example, we adjusted somewhat for this change, by "recalibrating" R' periodically. This gives us a better estimate. Were we to recalibrate more frequently, by using a stepsize Δt smaller than $1/3$, we would obtain still better results.

2. In the above example, we proceeded according to a definite algorithm:

   (a) We used our "current" value of R (initially, $R(0)$), and our rate equation (1.1.3), to compute a "current" value of R';

   (b) We used (PEGV), and the results of step (a) above, to compute the (approximate) change ΔR in R, over the interval in question;

   (c) We added this change ΔR to our "current" value of R, to get our "new" value of R;

(d) We returned to step (a) above and, letting our “new” value of R now play the role of the “current” value of R, repeated the process.

We “exited” the above iterative procedure once we reached the desired value of t.

3. We employed essentially the same algorithm, but with different quantities and numbers, in Example 1.1.4 above. This incredibly powerful algorithm is known as *Euler’s* (pronounced “Oiler’s”) *method*, after Leonhard Euler (1707-1783). We’ll make repeated use of this method – and of technology, to make the computations less onerous – throughout this text.

In the next section, we expand on Example 1.1.5 above, by focussing more carefully on three separate subpopulations – of susceptible, infected, and recovered individuals. We develop a rate equation for each of these subpopulations, and show how Euler’s method may then be used to chart the progression of the epidemic thus described.