MATH 4510: Review for Final Exam

1. Let U be a uniform random variable on the interval $(1,3)$. Let $X = \ln(U)$. Find the probability density function of X and compute the probability $P(X \leq 1)$.

   We compute $F_X(x) = P(X \leq x) = P(\ln U \leq x) = P(U \leq e^x) = \frac{e^x - 1}{3-1}$ for $0 \leq x \leq \ln 3$. So, the probability density function of X is

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) = \begin{cases} \frac{e^x}{2} & 0 < x < \ln 3 \\ 0 & \text{otherwise} \end{cases}.$$

   So, $P(X \leq 1) = \int_0^1 f(x)\mathrm{d}x = \frac{e^x}{2}\Big|_0^1 = \frac{e-1}{2}$.

2. The random variables X and Y are jointly continuous with joint probability density function $f(x,y) = cxy$ for $0 < x, y < 1$ and $f(x,y) = 0$ otherwise. Find the constant c and compute the marginal probability density functions f_X and f_Y. Find the conditional probability density function of X given that $Y = y$. Find $P(XY \leq 0.5)$.

   We have that $1 = \int_0^1 \int_0^1 cxy \, \mathrm{d}x\mathrm{d}y = \frac{c}{4}$, so that $c = 4$. We compute:

$$f_X(x) = \int_0^1 4xy\mathrm{d}y = 2x \text{ for } 0 < x < 1$$
$$f_Y(y) = \int_0^1 4xy\mathrm{d}x = 2y \text{ for } 0 < y < 1, \text{ and}$$
$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{4xy}{2y} = 2x \text{ for } 0 < x < 1.$$

   Finally,

$$P(XY \leq 0.5) = \int\int_{xy \leq 0.5} 4xy \, \mathrm{d}A = \int_0^{0.5}\int_0^1 4xy \, \mathrm{d}y\mathrm{d}x + \int_{0.5}^1 \int_0^{\frac{1}{2x}} 4xy \, \mathrm{d}y\mathrm{d}x$$
$$= \int_0^{0.5} 2x \, \mathrm{d}x + \int_{0.5}^1 \frac{2x}{4x^2}\mathrm{d}x = x^2\Big|_0^{0.5} + \frac{\ln x}{2}\Big|_{0.5}^1 = \frac{1}{4} + \frac{\ln 2}{2}.$$

   Aside: We see that $f(x,y) = f_X(x)f_Y(y)$, so that X and Y are independent, so we could have instead computed $f_{X|Y}$ by noting that $f_{X|Y}(x|y) = f_X(x)$ when X and Y are independent.

3. A coin that lands heads with probability p is flipped n times. Let X be the number of heads and Y be the number of tails. Compute the correlation coefficient $\rho(X,Y)$.

   We have that X and Y are both binomial with parameters n and p, so if we let $\sigma^2 = np(1-p)$, then $\operatorname{var}(X) = \operatorname{var}(Y) = \sigma^2$. Also, $X + Y = n$, so

$$\operatorname{cov}(X,Y) = \operatorname{cov}(X, n - X) = \operatorname{cov}(X,n) - \operatorname{cov}(X,X) = -\operatorname{var}(X) = -\sigma^2.$$

   Then,

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} = \frac{-\sigma^2}{\sigma^2} = -1.$$

Note that this is a special case of the fact that $\rho(X, Y) = \pm 1$ if and only if $Y = aX + b$ for some real numbers a and b with $a \neq 0$.

4. A fair die is rolled 7 times. Compute the expected number of different sides that the die will land on.

   Let $X_i = 1$ if the die lands on side i at least once and 0 otherwise. Let X be the number of different sides that the die lands on, so that $X = X_1 + \cdots + X_6$. We compute

   $$E(X_i) = P(X_i = 1) = 1 - P(\text{no roll lands on } i) = 1 - \left(\frac{5}{6}\right)^7.$$

   Then, $E(X) = \sum_{i=1}^{6} E(X_i) = 6\left(1 - \left(\frac{5}{6}\right)^7\right) = 4.326$.

5. Alex plans to complete ten tasks. She expects that she will spend an average of 12 minutes doing each task with a standard deviation of 4 minutes, independently of how long she takes on other tasks. Approximate the probability that it will take her no longer than 150 minutes to complete all of the tasks.

   Let X_i be the amount of time that Alex spends on the ith task and $X = X_1 + \cdots + X_{10}$ be the amount of time that Alex spends on all of the tasks combined. Then, by the Central Limit Theorem,

   $$P(X \leq 150) = P\left(\frac{X_1 + \cdots + X_{10} - 10 \cdot 12}{4 \cdot \sqrt{10}} \leq \frac{150 - 10 \cdot 12}{4 \cdot \sqrt{10}}\right) \approx P(Z \leq 2.37) = \Phi(2.37) = 0.9911.$$