MATH 4510: Review for Midterm Exam

1. A die is rolled 5 times. How many different sequences of rolls are possible if:

   (a) the die never lands on 4?

   There are five sides other than 4, so by the basic principle of counting, there are $5^5 = 3125$ different sequences.

   (b) the die lands on one number at least twice?

   There are 6^5 different sequences, of which $\binom{6}{5} \cdot 5!$ don't repeat any number, so there are $6^5 - \binom{6}{5} \cdot 5! = 7056$ different sequences with at least one number repeated.

2. Alice, Bob, and Carl each attempt to solve a crossword puzzle. There is a 70% chance that Alice can solve the puzzle without making a mistake, a 60% chance that Bob can, and a 85% chance that Carl can. What is the probability that at least one of them will solve the puzzle without making a mistake?

   The simplest way to solve this is to note that

   $$P(\text{each one makes a mistake}) = (1 - 0.7)(1 - 0.6)(1 - 0.85) = 0.018,$$

   so $P(\text{at least one doesn't make a mistake}) = 1 - P(\text{each one makes a mistake}) = 0.982$. Alternatively, this can be done using Inclusion-Exclusion.

3. In a certain area, 4% of men and 0.5% of women are colorblind. If a given person is colorblind, what is the probability that that person is a man? (You may assume that the number of men and women in the area are equal.)

   Let M be the event that the person is a man and C be the event that the person is colorblind. We will use Bayes' Rule in odds form. The prior odds are $\frac{P(M)}{P(M^c)} = 1$ and the likelihood ratio is $\frac{P(C|M)}{P(C|M^c)} = \frac{0.04}{0.005} = 8$. So, the posterior odds are $\frac{P(M|C)}{P(M|C^c)} = 8 \cdot 1 = 8$. Then, $P(M|C) = \frac{8}{8+1} = 0.89$.

4. A random variable X takes on the values -1, 3, and 5 with probabilities 0.1, 0.6, and 0.3 respectively. Find the probability mass function and cumulative distribution function of X. Compute $E(X)$ and $\text{var}(X)$.

   The probability mass function of X is given by $p(-1) = 0.1$, $p(3) = 0.6$, $p(5) = 0.3$, and $p(x) = 0$ for all x other than -1, 3, and 5. The cumulative distribution function of X is given by

   $$F(x) = \begin{cases} 0 & x < -1 \\ 0.1 & -1 \leq x < 3 \\ 0.7 & 3 \leq x < 5 \\ 1 & x \geq 5 \end{cases}.$$

We compute

$$E(X) = -1 \cdot 0.1 + 3 \cdot 0.6 + 5 \cdot 0.3 = 3.2 \text{ and}$$
$$\text{var}(X) = E(X^2) - E(X)^2 = \big((-1)^2 \cdot 0.1 + 3^2 \cdot 0.6 + 5^2 \cdot 0.3\big) - (3.2)^2 = 2.76.$$

5. You roll a fair die twelve times. Find the probability that the die will land on six at least twice.

   Let X be the number of rolls that land on six. Each roll independently has a probability of $\frac{1}{6}$ to land on six, so $X \sim B\left(12, \frac{1}{6}\right)$. Thus,

$$P(X = k) = \binom{12}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{12-k}.$$

   We compute:

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \left(\frac{5}{6}\right)^{12} - 12 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{11} = 0.6186.$$