MATH 4510: Review for Final Exam

1. A die is rolled 5 times. How many different sequences of rolls are possible if:

   (a) the die never lands on 4?

   There are five sides other than 4, so by the basic principle of counting, there are $5^5 = 3125$ different sequences.

   (b) the die lands on one number at least twice?

   There are 6^5 different sequences, of which $\binom{6}{5} \cdot 5!$ don't repeat any number, so there are $6^5 - \binom{6}{5} \cdot 5! = 7056$ different sequences with at least one number repeated.

2. Alice, Bob, and Carl each attempt to solve a crossword puzzle. There is a 70% chance that Alice can solve the puzzle without making a mistake, a 60% chance that Bob can, and a 85% chance that Carl can. What is the probability that at least one of them will solve the puzzle without making a mistake?

   The simplest way to solve this is to note that

$$P(\text{each one makes a mistake}) = (1 - 0.7)(1 - 0.6)(1 - 0.85) = 0.018,$$

   so $P(\text{at least one doesn't make a mistake}) = 1 - P(\text{each one makes a mistake}) = 0.982$. Alternatively, this can be done using Inclusion-Exclusion.

3. In a certain area, 4% of men and 0.5% of women are colorblind. If a given person is colorblind, what is the probability that that person is a man? (You may assume that the number of men and women in the area are equal.)

   Let M be the event that the person is a man and C be the event that the person is colorblind. We will use Bayes' Rule in odds form. The prior odds are $\frac{P(M)}{P(M^c)} = 1$ and the likelihood ratio is $\frac{P(C|M)}{P(C|M^c)} = \frac{0.04}{0.005} = 8$. So, the posterior odds are $\frac{P(M|C)}{P(M|C^c)} = 8 \cdot 1 = 8$. Then, $P(M|C) = \frac{8}{8+1} = 0.89$.

4. A random variable X takes on the values -1, 3, and 5 with probabilities 0.1, 0.6, and 0.3 respectively. Find the probability mass function and cumulative distribution function of X. Compute $E(X)$ and $\text{var}(X)$.

   The probability mass function of X is given by $p(-1) = 0.1$, $p(3) = 0.6$, $p(5) = 0.3$, and $p(x) = 0$ for all x other than -1, 3, and 5. The cumulative distribution function of X is given by

$$F(x) = \begin{cases} 0 & x < -1 \\ 0.1 & -1 \le x < 3 \\ 0.7 & 3 \le x < 5 \\ 1 & x \ge 5 \end{cases}.$$

We compute

$$E(X) = -1 \cdot 0.1 + 3 \cdot 0.6 + 5 \cdot 0.3 = 3.2 \text{ and}$$
$$\mathrm{var}(X) = E(X^2) - E(X)^2 = \big((-1)^2 \cdot 0.1 + 3^2 \cdot 0.6 + 5^2 \cdot 0.3\big) - (3.2)^2 = 2.76.$$

5. Let U be a uniform random variable on the interval $(1, 3)$. Let $X = \ln(U)$. Find the probability density function of X and compute the probability $P(X \leq 1)$.

   We compute $F_X(x) = P(X \leq x) = P(\ln U \leq x) = P(U \leq e^x) = \frac{e^x - 1}{3-1}$ for $0 \leq x \leq \ln 3$. So, the probability density function of X is

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x) = \begin{cases} \frac{e^x}{2} & 0 < x < \ln 3 \\ 0 & \text{otherwise} \end{cases}.$$

   So, $P(X \leq 1) = \int_0^1 f(x)\mathrm{d}x = \frac{e^x}{2}\Big|_0^1 = \frac{e-1}{2}$.

6. The random variables X and Y are jointly continuous with joint probability density function $f(x, y) = cxy$ for $0 < x, y < 1$ and $f(x, y) = 0$ otherwise. Find the constant c and compute the marginal probability density functions f_X and f_Y. Find the conditional probability density function of X given that $Y = y$. Find $P(XY \leq 0.5)$.

   We have that $1 = \int_0^1 \int_0^1 cxy \, \mathrm{d}x\mathrm{d}y = \frac{c}{4}$, so that $c = 4$. We compute:

$$f_X(x) = \int_0^1 4xy\mathrm{d}y = 2x \text{ for } 0 < x < 1$$
$$f_Y(y) = \int_0^1 4xy\mathrm{d}x = 2y \text{ for } 0 < y < 1, \text{ and}$$
$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{4xy}{2y} = 2x \text{ for } 0 < x < 1.$$

   Finally,

$$P(XY \leq 0.5) = \int\int_{xy \leq 0.5} 4xy \, \mathrm{d}A = \int_0^{0.5} \int_0^1 4xy \, \mathrm{d}y\mathrm{d}x + \int_{0.5}^1 \int_0^{\frac{1}{2x}} 4xy \, \mathrm{d}y\mathrm{d}x$$
$$= \int_0^{0.5} 2x \, \mathrm{d}x + \int_{0.5}^1 \frac{2x}{4x^2}\mathrm{d}x = x^2\big|_0^{0.5} + \frac{\ln x}{2}\big|_{0.5}^1 = \frac{1}{4} + \frac{\ln 2}{2}.$$

   Aside: We see that $f(x, y) = f_X(x)f_Y(y)$, so that X and Y are independent, so we could have instead computed $f_{X|Y}$ by noting that $f_{X|Y}(x|y) = f_X(x)$ when X and Y are independent.

7. A coin that lands heads with probability p is flipped n times. Let X be the number of heads and Y be the number of tails. Compute the correlation coefficient $\rho(X, Y)$.

We have that X and Y are both binomial with parameters n and p, so if we let $\sigma^2 = np(1-p)$, then $\operatorname{var}(X) = \operatorname{var}(Y) = \sigma^2$. Also, $X + Y = n$, so

$$\operatorname{cov}(X, Y) = \operatorname{cov}(X, n - X) = \operatorname{cov}(X, n) - \operatorname{cov}(X, X) = -\operatorname{var}(X) = -\sigma^2.$$

Then,

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} = \frac{-\sigma^2}{\sigma^2} = -1.$$

Note that this is a special case of the fact that $\rho(X, Y) = \pm 1$ if and only if $Y = aX + b$ for some real numbers a and b with $a \neq 0$.

8. A fair die is rolled 7 times. Compute the expected number of different sides that the die will land on.

   Let $X_i = 1$ if the die lands on side i at least once and 0 otherwise. Let X be the number of different sides that the die lands on, so that $X = X_1 + \cdots + X_7$. We compute

   $$E(X_i) = P(X_i = 1) = 1 - P(\text{no roll lands on } i) = 1 - \left(\frac{5}{6}\right)^7.$$

   Then, $E(X) = \sum_{i=1}^{7} E(X_i) = 7\left(1 - \left(\frac{5}{6}\right)^7\right) = 5.046$.

9. In a good year, the number of winter storms is Poisson distributed with parameter 3. In a bad year, the number of winter storms is Poisson distributed with parameter 5. If the next year will be a good year with probability 0.4 or a bad year with probability 0.6, find the expected value and the variance of the number of storms that will occur.

   Let X be the number of winter storms next year and let $Y = 1$ if next year is a good year and $Y = 0$ otherwise. Then, $E(X|Y = 1) = 3$ and $E(X|Y = 0) = 5$. So,

   $$E(X) = E[E(X|Y)] = E(X|Y = 1)P(Y = 1) + E(X|Y = 0)P(Y = 0) = 3 \cdot 0.4 + 5 \cdot 0.6 = 4.2.$$

   Similarly, $\operatorname{var}(X|Y = 1) = 3$ and $\operatorname{var}(X|Y = 0) = 5$. We compute:

   $$\begin{aligned}
   \operatorname{var}[E(X|Y)] &= 3^2 \cdot 0.4 + 5^2 \cdot 0.6 = 18.6, \\
   E[\operatorname{var}(X|Y)] &= 3 \cdot 0.4 + 5 \cdot 0.6 = 4.2, \text{ and so} \\
   \operatorname{var}(X) &= E[\operatorname{var}(X|Y)] + \operatorname{var}[E(X|Y)] = 22.8.
   \end{aligned}$$

10. An exam will consist of ten questions. A student expects that she will spend an average of 12 minutes doing each question with a standard deviation of 4 minutes, independently of how long she takes on other questions. If she has 150 minutes to do the exam, approximate the probability that she will finish the exam in the allotted amount of time.

    Let X_i be the amount of time the student spends on the ith question and $X = X_1 + \cdots + X_{10}$

be the amount of time that the student spends on the exam. Then, by the Central Limit Theorem,

$$P(X \leq 150) = P\left(\frac{X_1 + \cdots + X_{10} - 10 \cdot 12}{4 \cdot \sqrt{10}} \leq \frac{150 - 10 \cdot 12}{4 \cdot \sqrt{10}}\right) \approx P(Z \leq 2.37) = \Phi(2.37) = 0.9911.$$