MATH 4510: Review for Exam 1 (Sections 1.1-3.2)

Answers in blue.

1. In how many different ways can a group of 12 people (6 men and 6 women) be divided into two groups of size 6 if each group has to have the same number of men and women?

   We need to choose 3 of the 6 men and 3 of the 6 women to be in the first group and put the remaining 3 men and 3 men in the second group. This can be done in $\binom{6}{3} \cdot \binom{6}{3}$ ways.

2. Suppose that N raccoons live in Boulder. A scientist traps 100 raccoons, tags them, and releases them back into the wild. A month later, another 50 raccoons are trapped. What is the probability that exactly k of them will be tagged? (Assume that the raccoon population did not change during this period and that all raccoons are equally likely to be trapped.)

   The sample space consists of all $\binom{N}{50}$ sets of 50 raccoons from the population of N that could be caught in the second month. As there are 100 tagged raccoons, there are $\binom{100}{k}$ ways to catch k tagged raccoons and $\binom{N-100}{50-k}$ ways to catch $50 - k$ un-tagged raccoons. So, the desired probability is

$$P(\text{catch exactly } k \text{ tagged raccoons}) = \frac{\binom{100}{k}\binom{N-100}{50-k}}{\binom{N}{50}}.$$

3. Let E, F, and G be events. Find (and prove) a formula for the probability that exactly one of the events E and F will occur and that G will not occur, in terms of $P(E)$, $P(F)$, $P(G)$, $P(EF)$, $P(EG)$, $P(FG)$, and $P(EFG)$. (You don't necessarily have to use all of these in the formula.)

   We wish to compute $P(EF^cG^c \cup E^cFG^c)$. Define

$$\begin{aligned} A &= EF^cG^c \\ B &= EFG^c \\ C &= E^cFG^c \\ D &= EF^cG \\ H &= EFG \\ I &= E^cFG. \end{aligned}$$

   (I suggest drawing a Venn diagram with these regions labeled to make this clearer.)
   We compute (using axiom 3 and the fact that these regions are all pairwise mutually exclusive)

that

$$\begin{aligned}
P(EF^cG^c \cup E^cFG^c) &= P(A) + P(C), \\
P(E) &= P(A) + P(B) + P(D) + P(H), \\
P(F) &= P(B) + P(C) + P(H) + P(I), \\
P(EF) &= P(B) + P(H), \\
P(EG) &= P(D) + P(H), \\
P(FG) &= P(H) + P(I), \text{ and} \\
P(EFG) &= P(H).
\end{aligned}$$

Thus,

$$\begin{aligned}
&P(E) + P(F) - 2P(EF) - P(EG) - P(FG) + 2P(EFG) \\
&\quad = (P(A) + P(B) + P(D) + P(H)) + (P(B) + P(C) + P(H) + P(I)) \\
&\qquad - 2(P(B) + P(H)) - (P(D) + P(H)) - (P(H) + P(I)) + 2P(H) \\
&\quad = P(A) + P(C) = P(EF^cG^c \cup E^cFG^c)
\end{aligned}$$

So, the desired formula is:

$$P(EF^cG^c \cup E^cFG^c) = P(E) + P(F) - 2P(EF) - P(EG) - P(FG) + 2P(EFG).$$

4. You roll two dice. If the sum of the dice is at least 9, what is the conditional probability that the sum is odd?

   Let E be the event that the sum is odd and F be the event that the sum is at least 9. Then EF is the event that the sum is 9 or 11. As all 36 outcomes are equally likely, we calculate that:

$$\begin{aligned}
P(EF) &= \frac{4}{36} + \frac{2}{36} = \frac{6}{36} \text{ and} \\
P(F) &= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36},
\end{aligned}$$

   where we have used the fact that the events for different sums are pairwise mutually exclusive and calculated $P(F)$ by adding the probabilities that the sum is 9, 10, 11, or 12. So,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{6/36}{10/36} = 0.6.$$

5. An integer is chosen at random from the numbers $1, \ldots, 1000$ with each integer equally likely to be chosen. What is the probability that the chosen integer is divisible by at least one of 2, 3, and 5? What is the probability that the chosen integer is divisible by none of 2, 3, and 5?

We will use the Principle of Inclusion-Exclusion. Let A be the event that the number chosen is divisible by 2, B be the event that it is divisible by 3, and C be the event that it is divisible by 5. Then AB is the event that the number is divisible by $2 \cdot 3 = 6$, AC is the event that the number is divisible by $2 \cdot 5 = 10$, and so on. So,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \\ &= \frac{500}{1000} + \frac{333}{1000} + \frac{200}{1000} - \frac{166}{1000} - \frac{100}{1000} - \frac{66}{1000} + \frac{33}{1000} \\ &= 0.734. \end{aligned}$$

Then, the probability that it is divisible by none of 2, 3, and 5 is

$$P((A \cup B \cup C)^c) = 1 - 0.734 = 0.266.$$

6. You are traveling from Denver to New York to Paris to London. You have one piece of luggage and at each stop it is transferred from one airplane to another. There is a 4% probability that a bag is lost at the Denver airport, a 12% probability that it is lost in New York, and a 6% probability that it is lost in Paris. What is the probability that your luggage will not reach London with you? If your luggage is lost, what is the conditional probability that it was lost in New York?

Let D be the event that your luggage is lost in Denver, N that it is lost in New York, and P that it is lost in Paris. Let $E = D \cup N \cup P$ be the event that your luggage does not reach London with you. Since your luggage can only get lost once, D, N, and P are pairwise mutually exclusive, so that $P(E) = P(D) + P(N) + P(P)$ by axiom 3. Note that $N = ND^c$ since the bag cannot be lost in New York unless it first successfully gets from Denver to New York. Similarly, $P = PD^cN^c$. We calculate:

$$\begin{aligned} &P(D) = 0.04, \\ &P(N) = P(ND^c) = P(D^c)P(N|D^c) = 0.96 \cdot 0.12 = 0.1152, \text{ and} \\ &P(P) = P(PD^cN^c) = P(D^c)P(N^c|D^c)P(P|D^cN^c) = 0.96 \cdot 0.88 \cdot 0.06 = 0.050688. \end{aligned}$$

So, $P(E) = 0.04 + 0.1152 + 0.050688 = 0.205888$. Then,

$$P(N|E) = \frac{P(NE)}{P(E)} = \frac{P(N)}{P(E)} = \frac{0.1152}{0.205888} \approx 0.5595.$$