Solutions for HW 5

All collected problems, even problems, and problems for which the solution in the back is overly brief are answered below. If you have a question on any others, please feel free to ask via e-mail or during my office hours. If you spot an error below, please let me know.

# Chapter 10

10.10 A point is chosen at random inside the unit circle. let the random variable $V$ denote the absolute value of the $x$-coordinate of the point. What is the expected value of $V$?

Let $X$ be the $x$-coordinate of the point. Then $F(v) = P(V \leq v) = P(|X| \leq v) = P(-v \leq X \leq v) = \frac{1}{\pi} \cdot$ (the area of the portion of the circle with $-v \leq x \leq v$). By symmetry, we can consider only the portion in the first quadrant and multiply by 4, so that

$$F(v) = \frac{4}{\pi} \int_0^v \sqrt{1 - x^2} \mathrm{d}x \text{ for } v \in [0, 1].$$

Taking a derivative, the probability density function is

$$f(x) = \begin{cases} \frac{4}{\pi}\sqrt{1-x^2} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

So, we calculate

$$E(V) = \int_0^1 \frac{4}{\pi} x\sqrt{1-x^2} \mathrm{d}x = -\frac{4}{3\pi}(1-x^2)^{3/2}\Big|_0^1 = \frac{4}{3\pi}.$$

10.13 A point $Q$ is chosen at random inside a sphere with radius $r$. What are the expected value and the standard deviation of the distance from the center of the sphere to the point $Q$?

We first calculate the cumulative distribution function for $X$ =the distance of $Q$ from the center of the sphere. Note that for $0 \leq x \leq r$,

$$P(X \leq x) = \frac{\text{volume of sphere of radius } x}{\text{volume of sphere of radius } r} = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi r^3} = \frac{x^3}{r^3}.$$

So, the probability density function of $X$ is $f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[P(X \leq x)] = \frac{3x^2}{r^3}$ for $0 < x < r$, and

$$f(x) = \begin{cases} \frac{3x^2}{r^3} & \text{if } 0 < x < r \\ 0 & \text{otherwise} \end{cases}.$$

Then, $E(X) = \int_0^r x f(x) \mathrm{d}x = \int_0^r x \cdot \frac{3x^2}{r^3} \mathrm{d}x = \frac{3r^4}{4r^3} = \frac{3}{4}r$ and $E(X^2) = \int_0^r x^2 \cdot \frac{3x^2}{r^3} \mathrm{d}x = \frac{3r^5}{5r^3} = \frac{3}{5}r^2$, so that $\mathrm{var}(X) = \frac{3}{5}r^2 - \left(\frac{3}{4}r\right)^2 = \frac{3}{80}r^2$ and $\sigma(X) = 0.1936r$.

# Chapter 4

4.36 Suppose that emergency response units are distributed throughout a large area according to a two-dimensional Poisson process. That is, the number of response units in any given bounded region has a Poisson distribution whose expected value is proportional to the area of the region, and the numbers of response units in disjoint regions are independent. An incident occurs at some arbitrary point. Argue that the probability of having at least one response unit within a distance $r$ is $1 - e^{-\alpha\pi r^2}$ for some constant $\alpha > 0$.

This is the analogue of the "time between arrivals" part of the time Poisson process. There is some constant $\alpha$ (the arrival intensity in the time case, and the density of responders here) such that the expected number of response units per square area is $\alpha$. Then

$$P(\text{at least one response unit within a distance } r) = 1 - P(\text{no response units within a distance } r).$$

If we let $T$ be the distance to the nearest responder, this is $1 - P(T \geq r) = 1 - e^{-\alpha\pi r^2}$ since the area within distance $r$ is $\pi r^2$ (since this region is a circle).