Solutions for HW 4

All collected problems, even problems, and problems for which the solution in the back is overly brief are answered below. If you have a question on any others, please feel free to ask via e-mail or during my office hours. If you spot an error below, please let me know.

Chapter 8

8.2 Someone has tossed a fair coin three times. You know that one of the tosses came up heads. What is the probability that at least one of the other two tosses came up heads as well?

Let X be the number of heads. Note $X \sim B(3, \frac{1}{2})$. Then $P(X \geq 2 | X \geq 1) = \frac{P(X\geq 2)}{P(X \geq 1)} = \frac{1}{2}$.

8.4 You travel from Amsterdam to Sidney with change of airplanes in Dubai and Singapore. You have one piece of luggage. At each stop your luggage is transferred from one airplane to another. At the airport in Amsterdam there is a probability of 5% that your luggage is not placed in the right plane. This probability is 3% at the airport in Dubai and 2% at the airport in Singapore. What is the probability that your luggage does not reach Sidney with you? If you luggage does not reach Sidney with you, what is the probability that it was lost at the airport of Dubai?

Let A, D, and S be the events that your luggage is lost in Amsterdam, Dubai, and Singapore respectively. Let $L = A \cup D \cup S$ be the event that your luggage is lost. Note that A, D, and S are mutually exclusive (since your luggage can only get lost once). Note $P(A) = 0.05$, $P(D) = 0.95 \cdot 0.03$ (as your luggage has to first not be lost in Amsterdam in order to get lost in Dubai), and $P(S) = 0.95 \cdot 0.97 \cdot 0.02$. Then $P(L) = P(A) + P(D) + P(S) = 0.097$.

Then, $P(D|L) = \frac{P(DL)}{P(L)} = \frac{P(D)}{P(L)} = \frac{0.95 \cdot 0.03}{0.097} = 0.294$.

8.8 A die is rolled to yield a number between 1 and 6, and then a coin is tossed that many times. What is the probability that heads will not appear?

Let $B_1, \ldots, B_6$ be the events that the die lands $1, \ldots, 6$ (respectively). Let A be the event that no heads appears. We use the law of conditional probabilities: $P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_6)P(B_6) = \frac{1}{6}(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^6}) = \frac{1}{6}\frac{1}{2}\frac{1-\frac{1}{2}^6}{1-\frac{1}{2}} = 0.1641$ (where we've used the Geometric Series Formula to evaluate the sum).

8.14 It is believed that a sought-after wreck will be in a certain sea area with probability $p = 0.4$. A search in that area will detect the wreck with probability $d = 0.9$ if it is there. What is the revised probability of the wreck being in the area when the area is searched and no wreck is found?

Let H be the event that the wreck is in the area and E be the event that the wreck is not found. Then $P(E) = P(E|H)P(H) + P(E|H^C)P(H^C) = (1-d)p + 1 \cdot (1-p)$ and $P(H|E) = \frac{P(HE)}{P(E)} = \frac{P(E|H)P(H)}{P(E)} = \frac{(1-d)p}{(1-d)p+(1-p)} = 0.0625$.

8.18 A friendly couple tells you that they did a 100% reliable sonogram test and found out that they are going to have twin boys. They asked the doctor about the probability of identical twins rather than fraternal twins. The doctor could only give them the information that the population proportion of identical twins is one-third (identical twins are always of the same sex but fraternal twins are random). Can you give the probability the couple asked for?

Let H be the event that the twins are identical and $\overline{H}$ be the event that the twins are fraternal. Let E be the evidence that the twins are of the same gender (we could also let E be the event that the twins are both boys; both result in the same answer). Then $\frac{P(H|E)}{P(\overline{H}|E)} = \frac{1}{\frac{1}{2}} \frac{\frac{1}{3}}{\frac{2}{3}} = 1$. The odds are $1:1$, so the probability is $\frac{1}{1+1} = 0.5$.

Chapter 9

9.3 You spin a game board spinner with 1,000 equal sections numbered as $1, 2, \ldots, 1000$. After your first spin, you have to decide whether to spin the spinner for a second time. Your payoff is the total score of your spins as long as this score does not exceed 1,000; otherwise, your payoff is zero. What strategy maximizes the expected value of your payoff?

The simplest way to do this is to focus just on the second spin. Suppose that we got a on the first spin. Now, look at the expected value of the second spin: if it's positive, then we want to make a second spin; if it's negative, then we don't want to make a second spin. If the second spin gives a number b such that $a + b \leq 1000$, then the second spin has value b. Otherwise, the second spin has value $-a$ (since we lose the a we would have gotten if we hadn't made a second spin). We calculate:

$$E(X) = \sum_{k=1}^{1000-a} b \cdot \frac{1}{1000} - a \cdot \frac{a}{1000} = \frac{(1000-a)(1001-a)}{2000} - \frac{a^2}{1000}.$$

Note this is a quadratic polynomial. Simplifying and setting it equal to zero, we see that it has the root 414.42. The parabola opens downward, so $E(X) \geq 0$ for $a \leq 414.42$ and $E(X) < 0$ for $a > 414.42$. Thus, the best strategy is to take a second spin if and only if the number on the first spin is 414 or smaller.

Another method is to decide that we'll take a second spin if the first is smaller than some number a, find the expected value of both spins in terms of a, and then use calculus maximization techniques to find the value of a for which the expected value is maximized. In this case, we let S_1 be the number on the first spin and S_2 be the number on the second spin (if we spin a second time). Then the payoff is

$$X = \begin{cases} S_1 & \text{if } S_1 > a \\ S_1 + S_2 & \text{if } S_1 \leq a \text{ and } S_1 + S_2 \leq 1000 \\ 0 & \text{otherwise} \end{cases}.$$

Note that the mass points of X are $0, 2, 3 \ldots, 1000$ (as long as $a \neq 1$). We compute the probability mass function of X case by case.

First, suppose $x \leq a$. Then, if $X \leq x$, it must be the case that we made two spins, so $P(X = x) = P(\bigcup_{m=1}^{x-1} S_1 = m, S_2 = x - m) = \sum_{m=1}^{x-1} P(S_1 = m, S_2 = x - m) = \sum_{m=1}^{x-1} P(S_1 = m)P(S_2 = x - m) = \sum_{m=1}^{x-1} \frac{1}{1000^2} = \frac{x-1}{10^6}$.

Next, suppose that $x > a$. Then it could be the case that $S_1 > a$ (so that we made only one spin) or that $S_1 \le a$ (in which case we made two spins). We calculate $P(X = x) = P(S_1 = a) + P(S_1 \ne a \text{ and } X = x) = P(S_1 = x) + P(\bigcup_{m=1}^{a} S_1 = m, S_2 = x - m)$, where the bounds on the last sum come from the fact that we would only take a second spin if $S_1 \le a$. So, $P(X = x) = \frac{1}{1000} + \sum_{m=1}^{a} P(S_1 = m)P(S_2 = x - m) = \frac{1}{1000} + \frac{a}{1000^2}$.

We don't need to calculate $P(X = 0)$ since this mass point won't affect the expected value, but if we could find it if desired: $P(X = 0) = 1 - \sum_{k=2}^{1000} P(X = k)$.

Now, $E(X) = 0 \cdot P(X = 0) + \sum_{k=2}^{a} k \cdot (10^{-6}(k-1)) + \sum_{k=a+1}^{1000} k \cdot (10^{-3} + 10^{-6})$. Simplify this using the formulas $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. We end up with a polynomial; taking a derivative, we find the critical point 413.92 that corresponds to a local maximum (or, a global maximum if we restrict to $a \ge 0$), so that we can maximize the expected value of \$609.89 by taking $a = 414$. (This method has the added advantage that we can actually calculate the expected value using it.)

9.10 What is the expected value of the number of times that two adjacent letters are the same in a random permutation of the word Mississippi?

Note that "Mississippi" has 11 letters and so 10 pairs of adjacent letters. Call the letters $b_1, \ldots, b_{11}$. Let X_i equal 1 if $b_i = b_{i+1}$ and 0 otherwise. Then, if let X be the number of times that adjacent letters are the same in a random permutation of the word, then $X = X_1 + \cdots + X_{10}$. We calculate:

$$\begin{aligned}
E(X_i) &= P(X_i = 1) \\
&= P(b_i = b_{i+1}) \\
&= P(b_{i+1} = \text{"M"} | b_i = \text{"M"})P(b_i = \text{"M"}) + P(b_{i+1} = \text{"I"} | b_i = \text{"I"})P(b_i = \text{"I"}) \\
&\quad + P(b_{i+1} = \text{"S"} | b_i = \text{"S"})P(b_i = \text{"S"}) + P(b_{i+1} = \text{"P"} | b_i = \text{"P"})P(b_i = \text{"P"}) \\
&= 0 \cdot \frac{1}{11} + \frac{3}{10}\frac{4}{11} + \frac{3}{10}\frac{4}{11} + \frac{1}{10}\frac{2}{11} \\
&= \frac{26}{110}.
\end{aligned}$$

So, $E(X) = 10 \cdot \frac{26}{110} = \frac{26}{11}$.

9.13 Consider Example 9.3 again. What is the standard deviation of the number of trials required?

Note $E(X^2) = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p = p\frac{1+(1-p)}{(1-(1-p))^3} = \frac{2-p}{p^2}$, so $\text{var}(X) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2} = \frac{4}{9}$. Then $\sigma(X) = \sqrt{\text{var}(X)} = \frac{2}{3}$.