Solutions for HW 3

All collected problems, even problems, and problems for which the solution in the back is overly brief are answered below. If you have a question on any others, please feel free to ask via e-mail or during my office hours. If you spot an error below, please let me know.

Note: The problems in this chapter depend on chance trees, while are difficult to type. If you have a question about how to draw a tree on one of these, please ask me in person during office hours.

Chapter 6

6.2 On a table before you are two bowls containing red and white marbles; the first bowl contains seven red and three white marbles, and the second bowl contains 70 red and 30 white marbles. You are asked to select one of the two bowls, from which you will blindly draw two marbles (with no replacing of the marbles). You will receive a prize if at least one of the marbles you picked is white. In order to maximize your probability of winning a prize, do you choose the first bowl or the second bowl?

We calculate using a chance tree that: $P(\text{at least one white}) = 1 - P(\text{both red}) = 1 - \frac{7}{10}\frac{6}{9}$ in the first bowl and $P(\text{at least one white}) = 1 - \frac{70}{100}\frac{69}{99}$ in the second bowl. This probability is larger for the first bowl.

Note that the probabilities are the same if you only draw one marble. The key difference in the two marble case is that the probabilities change faster in the first bowl than they do in the second.

6.8 Passers-by are invited to take part in the following sidewalk betting game. Three cards are placed into a hat. One card is red on both sides, one is black on both sides, and one is red on one side and black on the other side. A participant is asked to pick a card out of the hat at random, taking care to keep just one side of the card visible. After having picked the card and having seen the color of the visible side of the card, the owner of the hat bets the participant equal odds that the other side of the card will be the same color as the one shown. Is this a fair bet?

Let A be the event that the visible side is a certain color (i.e., either red or black) and B be the event that the other side is the same color. Then we calculate $P(B|A) = \frac{2}{3}$ that the other side of the card is the same color, so the correct odds are $\frac{\frac{2}{3}}{1-\frac{2}{3}} : 1 = 2 : 1$, so the bet is not fair.

6.11 There are two taxicab companies in a particular city, "Yellow Cabs" and "White Cabs." Of all the cabs in the city, 85% are "Yellow Cabs" and 15% are "White Cabs." The issue of cab color has become relevant in a hit-and-run case before the courts in this city, in which witness testimony will be essential in determining the guilt or innocence of the cab driver in question. In order to test witness reliability, the courts have set up a test situation similar to the one occurring on the night of the hit-and-run accident. Results show that 80% of the participants in the test case correctly identified the cab color, whereas 20% of the participants identified the wrong company. If an eyewitness believed that she saw a "White Cab," what is the probability that the cabbie responsible is a "White Cabs" employee?

Let E be the event that the witness identified the car as a "White Cab" and H be the event that the car was a "White Cab." We want to compute $P(H|E)$. Using Bayes' Theorem,

$$P(H|E) = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|H^C)P(H^C)} = \frac{0.8 \cdot 0.15}{0.8 \cdot 0.15 + 0.2 \cdot 0.85} \approx 0.414.$$

Civil law in the United States requires a "preponderance of the evidence," meaning that a positive judgment requires that the evidence makes it more than 50% likely that the accused is a tortfeasor. (Criminal law requires that the evidence be "beyond a reasonable doubt," which is not typically given as an exact percentage but is widely held as being a much stricter standard than 50%.) As such, if this issue arose in a civil case under current United States law, the "White Cabs" company would not be held responsible for the accident. This problem has been discussed in serious legal journals, as neither of the obvious alternatives (holding "White Cabs" responsible or not holding "White Cabs" responsible) is very attractive: if they are not held responsible in cases like this, they have no incentive to avoid hit-and-run accidents; if they are held responsible, the odds are that they're being punished for something that they didn't do (and also the "Yellow Cabs" company has an inefficiently low incentive to avoid hit-and-run accidents).

6.16 A sum of money is placed in each of two envelopes. The amounts differ from one another, but you do not know what the values of the two amounts are. You do know that the values lie between two boundaries m and M with $0 < m < M$. You choose an envelope randomly. After inspecting its contents, you may switch envelopes. Set up a chance tree to verify that the following procedure will give you a probability of greater than $\frac{1}{2}$ of winding up with the envelope holding the most cash.

(a) Choose an envelope and look to see how much cash is inside.

(b) Pick a random number between m and M.

(c) If the number you drew is greater than the amount of cash in your envelope, you exchange the envelope. Otherwise, you keep the envelope you have.

Let $f(x) = \frac{M-x}{M-m}$. Note $0 \leq f(x) \leq 1$ for $m \leq x \leq M$, so we can think of $f(x)$ as a probability. In particular, given the procedure above, $f(x)$ is the probability that we will switch envelopes if the amount of cash inside the envelope is x.

Let a and b be the amounts of money in the envelopes. Without loss of generality, suppose $a < b$. Then we will end up with the envelope with the most cash if we either: (i) pick the envelope containing a and decide to switch or (ii) pick the envelope containing b and decide not to switch. The probability of one of these occurring is $P(\text{win}) = \frac{1}{2}f(a) + \frac{1}{2}(1 - f(b))$ (as there is a $\frac{1}{2}$ chance of picking each envelope, a $f(a)$ probability of switching if we pick a and a $(1 - f(b))$ probability of not switching if we pick b). This simplifies to $\frac{1}{2} + \frac{1}{2}(f(a) - f(b))$, which will be greater than $\frac{1}{2}$ if and only if $f(a) > f(b)$. But f is a decreasing function, so $f(a) > f(b)$ since $a < b$.

This problem can be generalized: suppose that instead of restricting the amounts of money to the interval (m, M), we look at two envelopes containing any number (so, in the interval $(-\infty, \infty)$) and desire to pick the larger number. We can do so with probability better than $\frac{1}{2}$ by following the same procedure as above, except with $f(x)$ defined as any decreasing function with the properties that $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 0$. One possibility is $f(x) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1} x$. More generally, if $F(x)$ is a cumulative distribution function for some random variable and F is not horizontal on any interval, then $f(x) = 1 - F(x)$ will work. Any choice of $F(x)$ will lead to a probability of more than 50% of choosing the larger, but some distributions will sometimes give probabilities very close to 50%. If you have other information about the distribution from which a and b are chosen, you can use it to choose a better $F(x)$ (in the sense that the probability of selecting the larger number will be bigger).

There is another problem in probability also known as the "Two Envelope Problem," which deals with a paradox involved in using expected values in cases where the sample space is not clearly defined. cf. `http://en.wikipedia.org/wiki/Two_envelope_problem`.

6.17 In a television game show, you can win 10,000 dollars by guessing the composition of red and white marbles contained in a nontransparent vase. The vase contains a very large number of marbles. You must guess whether the vase has twice as many red marbles as white ones, or whether it has twice as many white ones as red ones. Beforehand, both possibilities are equally likely to you. To help you guess, you are given a one-time opportunity of picking one, two, or three marbles out of the vase. This action, however, comes at the expense of the 10,000 dollar prize money. If you opt to choose one marble out of the vase, $750 will be subtracted from the $10,000 should you win. Two marbles will cost you $1,000 and three marbles will cost you $1,500. Set up a chance tree to determine which strategy will help you maximize your winnings.

In each case, there is some probability p that you win n dollars and probability $1-p$ that you win nothing, so the expected value is np. While you might also use other considerations to decide which strategy to employ, if (as the problem suggests) your goal is to maximize your winnings, you should pick the strategy that maximizes this expected value.

In each case, clearly you want to guess that the dominant color of marbles in the vase is the same as the dominant color in the test marbles that you drew, or to guess either "white" or "red" with equal probability if you either draw no marbles or if you draw two marbles and they happen to be one white and one red. Calculate the expected value of drawing 0,1,2, or 3 marbles in four cases:

If you draw 0 marbles, you have a $\frac{1}{2}$ chance of correctly guessing the dominant color, and so your expected winnings are $10000 \cdot \frac{1}{2} = 5000$.

If you draw 1 marble, you have a $\frac{2}{3}$ chance of correctly guessing the dominant color, and so your expected winnings are $9250 \cdot \frac{2}{3} = 6166.66$.

If you draw 2 marbles, you have a $\frac{2}{3}$ chance of correctly guessing the dominant color, and so your expected winnings are $9000 \cdot \frac{2}{3} = 6000$.

If you draw 3 marbles, you have a $\frac{20}{27}$ chance of correctly guessing the dominant color, and so your expected winnings are $8500 \cdot \frac{20}{27} = 6296.30$.

As such, the best strategy is to draw three marbles and guess the dominant color among them.

Note that the idea behind this problem is to model the value of obtaining evidence when there is a cost involved in obtaining that evidence. Depending on how costly that evidence is, it may or may not be the case that the ability to make a better decision based on the evidence is worth more than the cost of obtaining the evidence. This is one reason why it's useful to solve a problem with constants like "n" instead of using concrete numbers: most serious problems in the real world involve a trade-off in which we want to make one quantity as large as possible while keeping another quantity as small as possible–for example, in this case, keeping the number of marbles n drawn fairly small while making the probability $P(\text{win when drawing } n \text{ marbles})$ fairly large.