Solutions for HW 2

All collected problems, even problems, and problems for which the solution in the back is overly brief are answered below. If you have a question on any others, please feel free to ask via e-mail or during my office hours. If you spot an error below, please let me know.

Chapter 4

4.6 In 1989, American publication *Money Magazine* assessed the performance of 277 important mutual funds over the previous ten years. For each of those ten years they looked at which mutual funds performed better than the S&P index. Research showed that five of the 277 funds performed better than the S&P index for eight or more years. Verify that the expected value of the number of funds performing better than the S&P index for eight years or more is equal to 15.2 when the investment portfolios of each fund have been compiled by a blindfolded monkey throwing darts at the *Wall Street Journal*. Assume that each annual portfolio has a 50% probability of performing better than the S&P index.

First, we want to compute the probability of one fund doing better than the index in eight or more years out of 10. Let X be the number of years (out of 10) that a fund does better than the index. Then $X \sim B(10, \frac{1}{2})$ and the desired probability is $P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10) \approx 0.0546875$.

Now, let Y be the number of funds (out of 277) that do better in eight or more years out of 10. Then $Y \sim B(277, 0.0546875)$, so that $E(Y) = 277 \cdot 0.0546875 \approx 15.148$.

(Note that this is much higher than the actual number of funds that performed this well, which is why it's sometimes suggested that it'd be better to replace mutual fund managers by blindfolded monkeys. One might ask how likely it is for the funds to do this poorly by chance. This probability is $P(Y \leq 5) \approx 0.002059$, so the performance of these funds is so bad as to be statistically significant.)

4.28 Calculate a Poisson approximation for the probability that in a thoroughly shuffled deck of 52 playing cards, it will occur at least one time that two cards of the same face value will succeed one another in the deck (two aces, for example). In addition, make the same calculation for the probability of three cards of the same face value succeeding one another in the deck.

There are 51 pairs of adjacent cards in the deck. Since there are $\frac{3}{51}$ ways in which the second card of a pair can be of the same face value as the first, we can approximate this probability using a Poisson distribution X with $\lambda = 51 \cdot \frac{3}{51} = 3$. (Note that these trials are weakly dependent, as a card which appears in one place in the deck can't appear in a different place later on.) So, the probability that this will occur at least once is $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-3}$.

In the second case, there are 50 triples of three adjacent cards in the deck and there are $\frac{3}{51} \cdot \frac{2}{50}$ ways that the three can share the same face value. Calculating in the same way, the probability is $P(X \geq 1) = 1 - e^{-\frac{6}{51}}$.

Chapter 5

5.4 The cholesterol level for an adult male of a specific racial group is normally distributed with an expected value of 5.2 mmol/l and a standard deviation of 0.65 mmol/l. Which cholesterol level is exceeded by 5% of the population?

The level exceeded by 5% is $x_{0.95} = 5.2 + z_{0.95}0.65 = 5.2 + (1.6449)(0.65) \approx 6.27$ mmol/l.

5.8 You wish to invest in two funds, A and B, both having the same expected return. The returns of the funds are negatively correlated with correlation coefficient ρ_{AB}. The standard deviations of the returns on funds A and B are given by σ_A and σ_B. Demonstrate that you can achieve a portfolio with the lowest standard deviation by investing a fraction f of your money in fund A and a fraction $1-f$ in fund B, where the optimal fraction f is given by $\frac{\sigma_B^2 - \sigma_A\sigma_B\rho_{AB}}{\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB}}$.

Suppose you invest a fraction f in A and a fraction $1-f$ in B. We wish to choose f so as to minimize $\sigma(fA + (1-f)B)$ where A and B are random variables giving the return from funds A and B. Note that minimizing σ^2 will also minimize σ, but is easier to work with. We compute:

$$\begin{aligned}\sigma^2(fA+(1-f)B) &= \sigma^2(fA) + \sigma^2((1-f)B) + 2\operatorname{cov}(fA,(1-f)B)\\ &= f^2\sigma^2(A) + (1-f)^2\sigma^2(B) + 2f(1-f)\operatorname{cov}(A,B)\\ &= f^2\sigma_A^2 + (1-f)^2\sigma^2(B) + 2f(1-f)\sigma_A\sigma_B\rho_{AB}\\ &= (\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB})f^2 + (2\sigma_A\sigma_B\rho_{AB} - 2\sigma_B^2)f + \sigma_B^2.\end{aligned}$$

We may write this last polynomial as af^2+bf+c for some a,b, and c. Note that $a>0$ since $\rho_{AB}<0$, so this is a parabola in f opening upward and so will indeed have a minimum. Using Calc I minimization techniques (or the quadratic formula), we find that the minimum occurs at $f = -\frac{b}{2a} = \frac{\sigma_B^2-\sigma_A\sigma_B\rho_{AB}}{\sigma_A^2+\sigma_B^2-2\sigma_A\sigma_B\rho_{AB}}$ as claimed.

Finally, since we have the constraint that $0 \le f \le 1$, it remains to verify that this minimum really occurs in this interval (since we can't invest a negative amount in a fund, or invest more than 100% of what we're investing in a fund). Since $\rho_{AB} < 0$, it's easy to see that $f > 0$. Since $\sigma_A^2 > 0$, we likewise see that $f < 1$.

Note the importance of the assumption $\rho_{AB} < 0$ in the above: this means that the stocks tend to move in opposite directions, so that when one goes down the other tends to go up. This is what allows us to minimize the risk: if both stocks tended to move in the same direction, then investing in both would not (in general) reduce risk.

5.16 The owner of a casino in Las Vegas claims to have a perfectly balanced roulette wheel. A spin of a perfectly balanced wheel stops on red an average of 18 out of 38 times. A test consisting of 2,500 trials delivers 1,105 red finishes. If the wheel is perfectly balanced, is this result plausible? Use the normal distribution to answer this question.

Let X be the number of red finishes out of 2500 spins. Then $X \sim B(2500, \frac{18}{38})$. Note $E(X) \approx 1184$, so that number of red finishes looks suspiciously low. Since $np \ge 5$ and $n(1-p) \ge 5$, we can approximate X by a normal: $X \sim N(2500 \cdot \frac{18}{38}, 2500 \cdot \frac{18}{38}\frac{20}{38})$. So, $P(X \le 1105) =$

$P(\frac{X-1184.21}{24.96} \leq \frac{1105-1184.21}{24.96}) \approx P(Z \leq -3.17) \approx 0.00076$. This is a very small probability and so this result is not plausible if the wheel is perfectly balanced.