Solutions for HW 1

All collected problems, even problems, and problems for which the solution in the back is overly brief are answered below. If you have a question on any others, please feel free to ask via e-mail or during my office hours. If you spot an error below, please let me know.

Chapter 2

2.2 In the television program "Big Sisters," 12 candidates remain. The public chooses four candidates for the final round. Each candidate has an equal probability of being chosen. The Gotham Echo reckons that the local heroine, Stella Stone, has a probability of 38.5% of getting through to the final: they give her a $\frac{1}{12}$ probability of being chosen first, a $\frac{1}{11}$ probability of being chosen second, a $\frac{1}{10}$ probability of being chosen third, and a $\frac{1}{9}$ probability of being chosen fourth. Is this calculation correct?

The idea here is that $\frac{1}{12}+\frac{1}{11}+\frac{1}{10}+\frac{1}{9}\approx 0.385$ and that these events are disjoint, so that the probabilities can just be added together. It's true that the events are disjoint, but the probabilities calculated by the Echo are wrong. Note that this is a compound experiment, and to be chosen second a candidate must also not be chosen first, so we need to take this probability into account.

$$\begin{aligned}
P(\text{chosen first}) &= \frac{1}{12}\\
P(\text{chosen second}) &= \frac{11}{12}\cdot\frac{1}{11}=\frac{1}{12}\\
P(\text{chosen third}) &= \frac{11}{12}\cdot\frac{10}{11}\cdot\frac{1}{10}=\frac{1}{12}\\
P(\text{chosen fourth}) &= \frac{11}{12}\cdot\frac{10}{11}\cdot\frac{9}{10}\cdot\frac{1}{9}=\frac{1}{12}
\end{aligned}$$

So, the correct probability is $\frac{1}{12}+\frac{1}{12}+\frac{1}{12}+\frac{1}{12}=\frac{1}{3}$.

Another way to do this calculation is to instead let the sample space consist of the $\binom{12}{4}$ unordered tuples of four candidates. There are $\binom{11}{3}$ ways to choose three candidates in addition to Stella to be chosen, so that $P(\text{Stella chosen})=\frac{\binom{11}{3}}{\binom{12}{4}}=\frac{1}{3}$.

2.4 (a) In Leakwater township, there are two plumbers. On a particular day three Leakwater residents call village plumbers independently of each other. Each resident randomly chooses one of the two plumbers. What is the probability that all three residents will choose the same plumber?

Let the first resident choose a plumber. Then we're interested in the probability that the second and third residents also choose that plumber. Each has a $\frac{1}{2}$ probability to do so and these are independent, so the probability is $\frac{1}{4}$. Alternatively, consider the 8 possible tuples of choices of plumbers. Of them, we want to consider the outcomes $(1,1,1)$ and $(2,2,2)$, corresponding to all three choosing the first or the second plumber, respectively. As each outcome is equally likely, the probability is $\frac{2}{8} = \frac{1}{4}$. This problem can also be solved using a $B(3, 0.5)$ distribution.

(b) You roll a fair die three times in a row. What is the probability that the second roll will deliver a higher point count than the first roll and the third roll a higher count than the second?

The direct way to do this is to note that there are $6^3 = 216$ possible outcomes and that the ones we're interested in are $(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,3,4), (1,3,5), (1,3,6), (1,4,5), (1,4,6), (1,5,6), (2,3,4), (2,3,5), (2,3,6), (2,4,5), (2,4,6), (2,5,6), (3,4,5), (3,4,6), (3,5,6), (4,5,6)$. This is twenty outcomes and each is equally likely, so the probability is $\frac{20}{216}$. The cleverer way to do this is to note that there are $6 \cdot 5 \cdot 4$ ways to choose three different numbers and that three numbers can be ordered in $3! = 6$ ways, of which only one is ascending order, so that there is a $\frac{\frac{1}{6} \cdot 6 \cdot 5 \cdot 4}{216}$ probability.

(c) Two players A and B each roll one die. The absolute difference of the outcomes is computed. Player A wins if the difference is $0, 1$, or 2; otherwise, player B wins. Is this a fair game?

Let X be the absolute difference of the outcomes. Then X is discrete with mass points 0,1,2,3,4,5. Looking at the $6^2 = 36$ possible outcomes, we compute that

$$P(X = 0) = \frac{6}{36}$$
$$P(X = 1) = \frac{10}{36}$$
$$P(X = 2) = \frac{8}{36}$$
$$P(X = 3) = \frac{6}{36}$$
$$P(X = 4) = \frac{4}{36}$$
$$P(X = 5) = \frac{2}{36}$$

(To verify that we didn't miss any, check that these probabilities sum to 1.) Then, $P(X \leq 2) = \frac{24}{36}$ while $P(X \geq 3) = \frac{12}{36}$, so the game isn't fair: player A is twice as likely to win as player B is.

2.7 In the dice game known as "seven," two fair dice are rolled and the sum of scores is counted. You bet on "manque" (that a sum of 2,3,4,5, or 6 will result) or on "passe" (that a sum of 8,9,10,11, or 12 will result). The sum of 7 is a fixed winner for the house. A winner receives a payoff that is double the amount staked on the game. Nonwinners forfeit the amount staked. Define an appropriate probability space for this experiment. Then calculate the expected value of the payoff per dollar staked.

The sample space is $\Omega = \{(i, j) \colon 1 \leq i, j \leq 6\}$. Each of the 36 outcomes is equally likely. There are 15 ways to roll a number less than 7 and 15 ways to roll a number greater than 7, so $P(\text{manque}) = P(\text{passe}) = \frac{15}{36}$. Let A be the event that the player bets on (that is, either manque or passe). Then the player receives \$2 if A occurs and receives nothing otherwise (with probability $1 - \frac{15}{36} = \frac{21}{36}$). So, if we let X be the amount of money the player receives if she stakes \$1, then $E(X) = \frac{15}{36} \cdot 2 + \frac{21}{36} \cdot 0 = \frac{30}{36}$. Note that the payoff to the player doesn't include the player's stake, so this game is favorable to the house.

2.13 The following game is played in a particular carnival tent. The carnival master has two covered beakers, each containing one die. He shakes the beakers thoroughly, removes the lids and peers inside. You have agreed that whenever at least one of the two dice shows an even number of points, you will bet with even odds that the other die will also show an even number of points. Is this a fair bet?

You can let Ω be the set of all 36 tuples of two numbers between 1 and 6, but since we only care about whether the dice are even or odd and the probabilities of these are equal, we can simplifying things by instead taking $\Omega = \{(E, E), (E, O), (O, E), (O, O)\}$. Note that the bet only occurs when at least one die is even, so we know that outcome (O, O) will not occur. Since this doesn't affect the probabilities of the other outcomes, we can take the probability of each remaining outcome to be $\frac{1}{3}$. (Note that, while correct here, this sort of reasoning is error-prone, as intuition often misleads us as to how removing outcomes affects the probabilities of those that remain. More on this in Chapter 6.) In the case (E, E), the master will show you either of the dice (with equal probability, say) to prove that one is even and you will make the bet and win, as the other is even too. In the cases (E, O) and (O, E), the master will show you the even die and you will make the bet and lose, as the other die is odd. So, you have a $\frac{1}{3}$ chance of winning this bet and it is not fair with even odds.

Intuition often fails in problems like this: we naturally assume that the probability of the die we don't get to see being even shouldn't be affected by the fact that the die we do get to see is even, but this is not correct in this circumstance because the master gets to check both dice before deciding which to show you. If we changed the problem so that the master only got to see the first die and then offered the bet if that die were even, then the bet would be fair (as the sample space would be $\Omega = \{(H, H), (H, T)\}$).

Chapter 7

7.6 Two people have agreed to meet at the train station between 12.00 and 1.00 p.m. Independently of one another, each person is to appear at a completely random moment between the hours of 12.00 and 1.00. What is the probability that the two persons will meet within 10 minutes of one another?

This problem can be done in either minutes or hours, as long as we're consistent. I'll do it in hours, so 10 minutes becomes $\frac{1}{6}$ hours. Let $\Omega = \{(x, y) : 0 \le x, y \le 1\}$. Let X be the absolute difference between the arrival times of the people, so $X = |x - y|$. Then, we want to compute $P(X \le \frac{1}{6}) = P(-\frac{1}{6} \le x - y \le \frac{1}{6})$. We can define probabilities as areas of the rectangle Ω, noting that the area of Ω is 1 so that we don't need to divide by anything. If you did this in minutes, you'd need to divide by 60^2. The region defined by $-\frac{1}{6} \le x - y \le \frac{1}{6}$ is bounded by the lines $y = x + \frac{1}{6}$ and $y = x - \frac{1}{6}$. Drawing these lines, we see that we want the area of the rectangle minus the area of the two triangles not included in our region. Each has area $\frac{1}{2}\left(\frac{5}{6}\right)^2$, so that the probability we want is $1 - \left(\frac{5}{6}\right)^2$.

7.7 The numbers B and C are chosen at random between -1 and 1, independently of each other. What is the probability that the quadratic equation $x^2 + Bx + C = 0$ has real roots. Also, derive a general expression for this probability when B and C are chosen at random from the interval $(-q, q)$ for $q > 0$.

We will solve for q. Then the case $q = 1$ can be found by substituting in this value. The discriminant of the quadratic is $B^2 - 4C$ and the quadratic will have real roots if and only if its discriminant is positive. That is, if $B^2 > 4C$. Let $\Omega = \{(B, C) : -q < B, C < q\}$. Interpreting probability as area in Ω divided by the area of Ω, the probability we want is the area above the quadratic $C = \frac{1}{4}B^2$ divided by $(2q)^2$. Call this event A. Since this isn't a simple geometric area, we'll want to find this by integrating. We need to break into the case $q < 4$ (in which case the parabola hits the left and right edges of the sample space) and the case $q \ge 4$ (in which case the parabola hits the upper edge of the sample space at $B = \pm 2\sqrt{q}$). In both cases, it'll be easier to first find $P(A^C)$.

Case $q < 4$**:** $P(A^C) = \frac{1}{4q^2}\int_{-q}^{q}(q - \frac{1}{4}B^2)\mathrm{d}B = \frac{1}{2} - \frac{q}{24}$. Thus, $P(A) = 1 - P(A^C) = \frac{1}{2} + \frac{q}{24}$.

Case $q \ge 4$**:** $P(A^C) = \frac{1}{4q^2}\int_{-2\sqrt{q}}^{2\sqrt{q}}(q - \frac{1}{4}B^2)\mathrm{d}B = \frac{2}{3\sqrt{q}}$, so $P(A) = 1 - \frac{2}{3\sqrt{q}}$.

7.10 Use the axioms to prove the following results:

(a) $P(A) \le P(B)$ if the set A is contained in the set B.

Since A is contained in B, we have $B = A \cup (B \setminus A)$, where A and $(B \setminus A)$ are disjoint. By Axiom 1, $P(B \setminus A) \ge 0$ and so $P(A) + P(B \setminus A) \ge P(A)$. But by Axiom 3 (or, the form of Axiom 3 in Rule 7.1), the left-hand side of this inequality is just $P(A \cup (B \setminus A)) = P(B)$, so we've shown that $P(B) \ge P(A)$.

(b) $P(\bigcup_{k=1}^{\infty} A_k) \le \sum_{k=1}^{\infty} P(A_k)$ for any sequence of subsets $A_1, A_2, \ldots$ (this result is known as *Boole's inequality*).

We first from this for $n = 2$. By the addition rule, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1A_2)$. By Axiom 1, $P(A_1A_2) \ge 0$ and hence $P(A_1 \cup A_2) \le P(A_1) + P(A_2)$.
Now, we proceed by induction. Assume that $P(\bigcup_{k=1}^{n} A_k) \le \sum_{k=1}^{n} P(A_k)$ for some $n \ge 2$. We want to show that $P(\bigcup_{k=1}^{n+1} A_k) \le \sum_{k=1}^{n+1} P(A_k)$. Let $B = \bigcup_{k=1}^{n} A_k$ so that $\bigcup_{k=1}^{n+1} A_k = B \cup A_{n+1}$. Then, using the $n = 2$ case and the induction hypothesis, we

have $P(\bigcup_{k=1}^{n+1} A_k) = P(B \cup A_{n+1}) \leq P(B) + P(A_{n+1}) = P(\bigcup_{k=1}^{n} A_k) + P(A_{n+1}) \leq \sum_{k=1}^{n} A_k + A_{n+1} = \sum_{k=1}^{n+1} A_k$.
So, we have $P(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} P(A_k)$ for all $n \geq 2$ by induction. We can now establish the desired result using the continuity property of probability:

$$P(\bigcup_{k=1}^{\infty} A_k) = P(\lim_{n\to\infty} \bigcup_{k=1}^{n} A_k) = \lim_{n\to\infty} P(\bigcup_{k=1}^{n} A_k) \leq \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k) = \sum_{k=1}^{\infty} P(A_k).$$

7.18 In the casino game of Chuck-a-Luck, three dice are contained within an hourglass-shaped, rotating cage. You bet on one of the six possible numbers and the cage is rotated. You lose money only if your number does not come up on any of the three dice. Much to the pleasure of the casinos, people sometimes reason as follows: the probability of my number coming up on one die is $\frac{1}{6}$ and so the probability of my number coming up on one of the three dice is $3 \times \frac{1}{6} = \frac{1}{2}$. Why is this reasoning false? How do you calculate the correct value of the probability that your number will come up on any of the three dice?

This reasoning is false since the event of your number coming up on one die is not disjoint from the event that your number comes up on one of the other dice. One easy way to see that this won't work is to imagine that their were seven dice instead of three: in that case, the probability of your number coming up would be greater than 1 if this reasoning were correct. (It is true, however, that the expected number of times your number will come up is $\frac{1}{2}$.)

One way to calculate this probability is to use the Principle of Inclusion-Exclusion. Let A_i be the probability that your number comes up on the ith die. Then $P(A_i) = \frac{1}{6}$, $P(A_iA_j) = \frac{1}{6^2}$, and $P(A_iA_jA_k) = \frac{1}{6^3}$ for any distinct i, j, k chosen from 1,2,3. So, by Inclusion-Exclusion, $P(\text{your number comes up at least once}) = P(A_1 \cup A_2) \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_2A_3) + P(A_1A_2A_3) = 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{36} + \frac{1}{216}$.

This problem can also be solved by letting X be the number of times that your number comes up and noting that $X \sim B(3, \frac{1}{6})$, in which case you want to compute $P(X \geq 1) = 1 - P(X = 0)$.

7.21 What is the probability that in a player's hand of 13 cards at least one suit will be missing?

Let A_i be the event that the ith suit is missing for $i = 1, 2, 3, 4$. Then we want to find $P(\bigcup_{k=1}^{4} A_k)$. We will use Inclusion-Exclusion. Note $P(A_i) = \frac{\binom{52-13}{13}}{\binom{52}{13}}$, as we want all 13 cards to be chosen from the 52-13 cards not in suit i. Similarly, $P(A_iA_j) = \frac{\binom{52-26}{13}}{\binom{52}{13}}$ and $P(A_iA_jA_k) = \frac{\binom{52-39}{13}}{\binom{52}{13}}$. Note $P(A_1A_2A_3A_4) = 0$, as it's not possible for a hand to contain none of the four suits. So, by Inclusion-Exclusion,

$$P(\bigcup_{k=1}^{4} A_k) = \sum_{k=1}^{3} (-1)^{k+1} \binom{4}{k} \frac{\binom{52-13k}{13}}{\binom{52}{13}} \approx 0.051.$$