Final Review for MATH 3510

This exam will cover Sections 6.1–6.2, 8.1–8.2, 9.1–9.5, 10.1, and 4.2.4. You will be permitted to use a calculator. There will be no proofs on the exam. As we've seen, calculating expected value and variance sometimes involve dealing with infinite sums or integrals; to keep things simple, any such sums on the exam will require knowledge only of the geometric series and its derivatives and any such integrals will involve nothing more complicated than polynomials. The exam will weigh the three chapters 6, 8, and 9 approximately equally and Sections 10.1 and 4.2.4 to a lesser extent.

1 Terminology, etc.

1. Be able to define: conditional probability, independence, expected value (for a discrete or continuous variable), probability density function, cumulative distribution function, Poisson process, memoryless property.
2. Be able to state: Bayes' rule (in probability and odds forms), law of conditional probabilities.
3. Be able to give the expected value, variance, probability density function, and cumulative distribution function for the exponential distribution (but I won't ask you to derive these).
4. Know that: $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $\dfrac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$, and $\dfrac{1+x}{(1-x)^3} = \sum_{k=1}^{\infty} k^2x^{k-1}$ (all for $|x| < 1$). I won't ask you these directly, but you may need to know them to do a given problem.

2 Calculation

5. Given a (fairly simple) probability mass function or probability density function of a random variable, you should be able to compute the expected value and variance of the variable. For example:
   (a) Let $0 < p < 1$. Let X be discrete with mass points $k = 1, 2, \ldots$ and probability mass function $P(X = k) = p(1-p)^{k-1}$.
   (In this case, we say that X has a *geometric distribution.*)
   (b) Let X be continuous with probability density function $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$.
   (In this case, we say that X has a *uniform distribution.*)
   (c) Let X be continuous with probability density function $f(x) = \begin{cases} 4x^3 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

3 Example Problems

6. I have 6 coins in my pocket: 3 are fair, one is double-headed, one is double-tailed, and one lands heads 75% of the time. I flip the coin three times and get heads each time. What is the probability that I am flipping the double-headed coin?

7. A point Q is chosen at random inside a sphere with radius r. What are the expected value and the standard deviation of the distance from the center of the sphere to the point Q?

8. Two gamblers, A and B, play the following game: each takes a turn drawing a card off the top of a well-shuffled deck until all 52 cards have been drawn. Every time a player draws a diamond, the other player gives the drawing player \$1. Let X be the number of dollars that A has won or lost after all 52 cards have been drawn. Find the expected value and standard deviation of X. (You may leave your answer unsimplified, if you wish.)

9. A bent coin comes up heads with probability p, where $0 < p < 1$.
   (a) You flip the coin until heads comes up. How many times do you expect to flip the coin?
   (b) You flip the coin until heads comes up r times. How many times do you expect to flip the coin?

10. Suppose a bridge player's hand of 13 cards contains an ace. What is the probability that the player has only one ace?

11. Let k be a constant and consider the function

$$f(x) = \begin{cases} kx^2 & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) For what value(s) of k is f a probability density function?
   (b) For the value(s) of k you found above, suppose X is a random variable with probability density function f. Find the expected value and standard deviation of X.

12. Suppose that U is a number chosen at random between 0 and 1 and that $X = \sqrt[3]{U}$. What is the expected value and standard deviation of X?

13. A hand of 13 cards is dealt from a well-shuffled deck. How many four-of-a-kinds do you expect the hand to contain? How many four-of-a-kinds do you expect the 39 cards remaining in the deck to contain?

14. A card is drawn from a deck of 52. Let A be the event that the card is a heart, B be the event that the card is red, and C be the event that the card is a king. Are events A and B independent? Are events A and C independent? (Justify your answer mathematically.)

15. An urn contains N balls, n of which are blue. If m balls are drawn from the urn, how many of the balls drawn do you expect to be blue?

16. A red die is rolled and the number that comes up is observed. Then, a blue die is rolled that many times. What is the probability that the blue die will come up 6 at least once?

17. A point Q is chosen at random from inside the triangle with vertices at $(0, 0)$, $(2, 2)$, and $(4, 0)$. Let X be the x-coordinate of Q. What are the expected value and standard deviation of X?

18. Suppose that the time between arrivals at a certain store is exponentially distributed with parameter $\lambda = 0.2$ minutes. If it's currently 11:00 am, when would the manager expect the next customer to enter? If it turns out that no customers enter between 11:00 am and 11:35 am, when would the manager expect the next customer to enter?

19. In a certain region, it rains on average once in every ten days during the summer. Rain is predicted on average for 85% of the days when rainfall actually occurs, while rain is predicted on average for 25% of the days when it does not rain. Assume that rain is predicted for tomorrow. What is the probability of rainfall actually occurring on that day?

4 Harder Problems (for those who find the above easy)

20. A fair coin is flipped $n+m$ times, landing heads n times and tails m times. After each flip, a "H" or a "T" is written down, so that afterward there is a record of the form "HHTHTHHTTTH." We call one or more flips coming up the same in a row a "run" (so that, for example, the record "HHTHTHHTTTH" contains 7 runs: "HH," "T," "H," "T," "HH," "TTT," and "H"). What is the expected number of runs? (*Hint:* Compute the number of heads-runs and tails-runs separately and add these together at the end. Note that the ith flip (for $i \neq 1$) begins a heads-run if the ith flip is a heads and the $(i-1)$st flip is a tails. The first flip begins a heads-run if it is a heads and begins a tails-run otherwise.)

21. You know that bowl A has three red and two white balls inside and that bowl B has four red balls and three white balls. Without your being aware of which one it is, one of the bowls is randomly chosen and presented to you. Blindfolded, you must pick two balls out of the bowl. You may proceed according to one of the following strategies:

    (a) you will choose and replace (*i.e.*, you will replace your first ball into the bowl before choosing your second ball).

    (b) you will choose two balls without replacing any (*i.e.*, you will not replace the first ball before choosing a second).

    The blindfold is then removed and the colors of both of the balls you chose are revealed to you. Thereafter you must make a guess as to which bowl your two balls came from. For each of the two possible strategies, determine how you can make your guess depending on the colors you have been shown. Which strategy offers the higher probability for a correct guess as to which bowl the balls came from?

22. Suppose that A flips a fair coin $n+1$ times and B flips a fair coin n times. What is the probability that A will have more heads than B? (*Hint:* Condition on who has more heads after the first n flips.)