MATH 3510: PROBABILITY AND STATS

July 1, 2011

FINAL EXAM

YOUR NAME: KEY: Answers in blue

Show all your work.

Answers out of the blue and without any supporting work may receive no credit *even if they are right!*

Write clearly. **Box your final answers.**

You may use a calculator. **No notes allowed.**

DO NOT WRITE IN THIS BOX!

problem	points	score
1	9 pts	
2	9 pts	
3	17 pts	
4	13 pts	
5	13 pts	
6	13 pts	
7	13 pts	
8	13 pts	
9	10 pts	
TOTAL	100 pts	

1. (9 pts) State the three properties of the Poisson process.

   (a) No arrivals are simultaneous.

   (b) The number of arrivals in disjoint time intervals are independent.

   (c) The number of arrivals in an interval is Poisson distributed with mean proportional to the length of the interval.

2. (9 pts) Define the following terms. Use full sentences and be as precise as possible.

   (a) cumulative distribution function

   The cdf of the random variable X is $F(x) = P(X \le x)$.

   (b) probability density function

   A pdf is a function f such that $f(x) \ge 0$ for all x and $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$.

   (c) conditional probability of A given B

   The conditional probability of A given B is $P(A|B) = \dfrac{P(AB)}{P(B)}$ if $P(B) > 0$.

3. (a) (5 pts) State Bayes' Rule (in either probability or odds form: your choice)

Let H and $\overline{H}$ be hypotheses and E be evidence. (That is, H, $\overline{H}$, and E are events.) Then

$$\frac{P(H|E)}{P(\overline{H}|E)} = \frac{P(E|H)}{P(E|\overline{H})}\frac{P(H)}{P(\overline{H})}.$$

(b) (12 pts) In a certain region, it rains on average once in every ten days during the summer. Rain is predicted on average for 85% of the days when rainfall actually occurs, while rain is predicted on average for 25% of the days when it does not rain. Assume that rain is predicted for tomorrow. What is the probability of rainfall actually occurring on that day?

Let H be the event that it rains, $\overline{H}$ be the event that it doesn't rain, and E be the event that rain is predicted. Then, by Bayes' Rule,

$$\frac{P(H|E)}{P(\overline{H}|E)} = \frac{0.85}{0.25}\frac{\frac{1}{10}}{\frac{9}{10}} = \frac{17}{45}.$$

Converting from odds to probability, $P(H|E) = 0.274$.

4. (13 pts) You flip a fair coin three times. Let A be the event that you get at least two heads. Let B be the event that not all of the coins come up the same (that is, you get at least one heads and at least one tails). Are A and B independent?

   Let X be the number of flips that land heads. Note $X \sim B(3, \frac{1}{2})$. We compute:

$$P(A) = P(X \geq 2) = \frac{4}{8} = \frac{1}{2},$$
$$P(B) = P(X \neq 0, X \neq 3) = \frac{6}{8} = \frac{3}{4}, \text{ and}$$
$$P(AB) = P(X = 2) = \frac{3}{8}.$$

   So, $P(AB) = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = P(A)P(B)$, so that the events A and B are independent.

5. (13 pts) Suppose an urn contains five red balls, three blue balls, and two green balls. You remove a ball at random from the urn (without replacing it). If the removed ball is red, you add two blue balls to the urn. If the removed ball is blue, you add two green balls to the urn. If the removed ball is green, you add two red balls to the urn. Then, you remove a second ball form the urn. What is the probability that the second ball you remove is red?

   Use the law of conditional probabilities. Let B_1, B_2, B_3 be the events that the first ball removed is red, blue, or green (respectively). Let A be the event that the second ball removed is red. Then

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) = \frac{4}{11}\frac{5}{10} + \frac{5}{11}\frac{3}{10} + \frac{7}{11}\frac{2}{10} \approx 0.4455.$$

6. (13 pts) A point Q is chosen at random inside a sphere with radius r. What is the expected value of the distance from the center of the sphere to the point Q?
Hint: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

We first calculate the cumulative distribution function for X =the distance of Q from the center of the sphere. Note that for $0 \leq x \leq r$,

$$P(X \leq x) = \frac{\text{volume of sphere of radius } x}{\text{volume of sphere of radius } r} = \frac{\frac{4}{3}\pi x^3}{\frac{4}{3}\pi r^3} = \frac{x^3}{r^3}.$$

So, the probability density function of X is $f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[P(X \leq x)] = \frac{3x^2}{r^3}$ for $0 < x < r$, and

$$f(x) = \begin{cases} \frac{3x^2}{r^3} & \text{if } 0 < x < r \\ 0 & \text{otherwise} \end{cases}.$$

Then, $E(X) = \int_0^r x f(x)\mathrm{d}x = \int_0^r x \cdot \frac{3x^2}{r^3}\mathrm{d}x = \frac{3r^4}{4r^3} = \frac{3}{4}r$.

7. (13 pts) Each day, a certain machine has a 0.01 probability of breaking. Let X be the number of days until the machine breaks. Find $E(X)$ and $\mathrm{var}(X)$.

Note that X has probability mass function $P(X = k) = (0.99)^{k-1}(0.01)$ (since it doesn't break the first $(k-1)$ days and does break on the kth day). So,

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} k(0.99)^{k-1}(0.01) = \frac{0.01}{(1-0.99)^2} = 100, \\ E(X^2) &= \sum_{k=1}^{\infty} k^2(0.99)^{k-1}(0.01) = \frac{0.01(1+0.99)}{(1-0.99)^3} = 19900, \text{ and so} \\ \mathrm{var}(X) &= 19900 - (100)^2 = 9900. \end{aligned}$$

Alternatively, you can assume that X has an exponential distribution. (This will only be approximately true, but I gave credit for solving the problem by this method.) Then $E(X) = 100$ and $\mathrm{var}(X) = 10000$.

8. (13 pts) An urn contains 20 blue balls and 30 red balls. You draw 10 balls from the urn. How many of the balls drawn do you expect to be blue?

   Let X be the number of blue balls drawn. Then $X = X_1 + \cdots + X_{10}$ where X_i is 1 if the ith ball drawn is blue and 0 otherwise. Then $E(X_i) = P(X_i = 1) = \frac{20}{50}$, so that $E(X) = 10 \cdot \frac{20}{50} = 4$.

   Alternatively, you can label the blue balls $b_1, \ldots, b_{20}$ and let $X = Y_1 + \cdots + Y_{20}$ where Y_i is 1 if the ball b_i is drawn and 0 otherwise.

9. **(Extra Credit, 10 pts)** You roll a die repeatedly until two consecutive rolls land on the same number. How many times do you expect to roll the die?

Let X be the number of times you roll the die until two consecutive rolls land on the same number. Note that in order for the kth roll to be the fist time this happens, it must be the case that no two rolls from the first $k-1$ were the same (with probability $\left(\frac{5}{6}\right)^{k-2}$, where we have $k-2$ instead of $k-1$ since the first roll can be anything) and that the kth roll is the same as the previous roll (with probability $\frac{1}{6}$), so that

$$P(X = k) = \frac{5}{6}^{k-2}\frac{1}{6} \text{ for } k = 2, 3, \ldots.$$

Then

$$\begin{aligned}
E(X) &= \sum_{k=2}^{\infty} k\left(\frac{5}{6}\right)^{k-2}\frac{1}{6} \\
&= \frac{1}{6}\sum_{k=1}^{\infty}(k+1)\left(\frac{5}{6}\right)^{k-1} \\
&= \frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1} + \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1} \\
&= \frac{1}{6}\frac{1}{(1-\frac{5}{6})^2} + \frac{1}{6}\frac{1}{1-\frac{5}{6}} \\
&= 6 + 1 \\
&= 7.
\end{aligned}$$