MATH 3510: PROBABILITY AND STATS

June 15, 2011

MIDTERM EXAM

YOUR NAME: KEY: Answers in Blue

Show all your work.

Answers out of the blue and without any supporting work may receive no credit *even if they are right!*

Write clearly. **Box your final answers.**

You may use a calculator. **No notes allowed.**

DO NOT WRITE IN THIS BOX!

problem	points	score
1	9 pts	
2	9 pts	
3	10 pts	
4	10 pts	
5	9 pts	
6	10 pts	
7	6 pts	
8	7 pts	
9	10 pts	
10	10 pts	
11	10 pts	
12	10 pts	
TOTAL	100 pts	

1. (9 pts) State the Kolmogorov axioms for a probability measure.

   1. For all events A, $P(A) \geq 0$.
   2. $P(\Omega) = 1$, where Ω denotes the sample space.
   3. For a sequence of mutually exclusive events $A_1, A_2, \ldots,$

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

2. (9 pts) Define the following terms. Use full sentences and be as precise as possible.

   (a) sample space

   The *sample space* is the set of all outcomes of a chance experiment.

   (b) event

   An *event* is a (nice) subset of the sample space.

   (c) random variable

   A *random variable* X is a function from the sample space Ω to the set of real numbers.

3. (10 pts) Recall that the covariance of random variables X and Y is defined to be

$$\operatorname{cov}(X, Y) = E[(X - E(X))(Y - E(Y))].$$

Prove that $\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y) + 2\operatorname{cov}(X, Y)$.
(If you wish, you may assume the fact that $E(aX + bY) = aE(X) + bE(Y)$ without proof.)

Proof: We compute

$$\begin{aligned}
\sigma^2(X + Y) &= E[(X + Y - E(X + Y))^2] = E[(X + Y - E(X) - E(Y))^2] \\
&= E[((X - E(X)) + (Y - E(Y)))^2] \\
&= E[(X - E(X))^2 + 2(X - E(X))(Y - E(Y)) + (Y - E(Y))^2] \\
&= E[(X - E(X))^2] + 2E[(X - E(X))(Y - E(Y))] + E[(Y - E(Y))^2] \\
&= \sigma^2(X) + \sigma^2(Y) + 2\operatorname{cov}(X, Y).
\end{aligned}$$

4. (10 pts) Let A and B be events where the set A is a subset of the set B. Show that $P(A) \leq P(B)$. You may use the Kolmogorov axioms and any other basic rules that we've seen in class. If a rule requires a certain hypothesis (such as two events being disjoint), be sure to explain why the hypothesis is satisfied.

Proof: Let $C = B \setminus A$. Since A is a subset of B, we have $B = A \cup C$ and A and C are disjoint. Since A and C are disjoint, we have by Rule 7.1 that $P(B) = P(A) + P(C)$. Also, by Axiom 1, $P(C) \geq 0$, so that $P(B) \geq P(A)$.

5. (9 pts) Roll two dice and let X be the larger of the numbers rolled.

(a) Find the probability mass function of X.

$$P(X=1)=\frac{1}{36}$$
$$P(X=2)=\frac{3}{36}$$
$$P(X=3)=\frac{5}{36}$$
$$P(X=4)=\frac{7}{36}$$
$$P(X=5)=\frac{9}{36}$$
$$P(X=6)=\frac{11}{36}$$

More compactly, $P(X=k)=\frac{2k-1}{36}$ for $k=1,\dots,6$.

(b) Find the expected value of X.

$E(X)=\sum_{k=1}^{6} kP(X=k)=\frac{161}{36}\approx 4.472.$

(c) Find the standard deviation of X.

$E(X^2)=\sum_{k=1}^{6} k^2P(X=k)=\frac{791}{36}$, so $\text{var}(X)=\frac{791}{36}-\left(\frac{161}{36}\right)^2=\frac{2555}{1296}\approx 1.971$. Thus, $\sigma(X)=\sqrt{\text{var}(X)}\approx 1.404$.

6. (10 pts) In the problem below, be sure to include any necessary parameters (n, p, λ, etc.)

   (a) Find the expected value and variance for a Bernoulli distribution (that is, a random variable that equals 1 on the success of a Bernoulli experiment and 0 otherwise).

   Let I be Bernoulli with paramater p. Then $E(I) = 1 \cdot p + 0 \cdot (1-p) = p$ and $\text{var}(I) = E(I^2) - E(I)^2 = (1^2 \cdot p + 0^2 \cdot (1-p)) - p^2 = p - p^2 = p(1-p)$.

   (b) Find the expected value and variance for a binomial distribution. (You probably want to use your answers from (a).)

   Let X be binomial with parameters n and p. Write $X = I_1 + \dots I_n$ where $I_1, \dots, I_n$ are independent Bernoulli trials with parameter p. Then $E(X) = E(I_1) + \cdots + E(I_n) = np$ and $\text{var}(X) = \text{var}(I_1) + \cdots + \text{var}(I_n) = np(1-p)$.

7. (6 pts) Let A and B be events with $P(A) = 0.8$ and $P(B) = 0.4$. If the probability that both A and B occur is 0.3, what is the probability that neither A nor B occurs?

   $P(A^C B^C) = P((A \cup B)^C) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(AB)) = 1 - (0.8 + 0.4 - 0.3) = 0.1.$

8. (7 pts) Suppose the random variables $X_1, \ldots, X_n$ are independent and identically distributed with expected value μ and standard deviation σ. Find the expected value and standard deviation of the random variable $X = X_1 + \cdots + X_n$ in terms of μ and σ. If n is large, what can you say about the distribution of the variable X?

$E(X) = E(X_1) + \cdots + E(X_n) = \mu + \cdots + \mu = n\mu.$
$\sigma(X) = \sqrt{\text{var}(X)} = \sqrt{\text{var}(X_1) + \cdots + \text{var}(X_n)} = \sqrt{\sigma^2 + \cdots + \sigma^2} = \sigma\sqrt{n}.$

For n large, the Central Limit Theorem tells us that X will have an approximately normal distribution: $X \sim N(n\mu, \sigma^2 n)$.

9. (10 pts) What are the chances of getting at least three sixes in one throw of 18 dice?

Let X be the number of sixes in one throw of 18 dice. Then $X \sim B(18, \frac{1}{6})$.

Then

$$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 2) \\ &= 1 - P(X = 2) - P(X = 1) - P(X = 0) \\ &= 1 - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} - \binom{18}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{17} - \binom{18}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{18} \\ &\approx 0.597. \end{aligned}$$

10. (10 pts) What is the probability that a hand of 13 cards contains four of a kind?

We will use the Principle of Inclusion-Exclusion. For the 13 ranks, let A_i be the event that the hand contains four cards of rank i, $1 \leq i \leq 13$.

We compute $P(A_i) = \frac{\binom{48}{9}}{\binom{52}{5}}$ (as four of the thirteen cards in the hand are fixed, so that the other 9 must be chosen from the 48 that remain in the deck) and similarly that $P(A_i A_j) = \frac{\binom{44}{5}}{\binom{52}{5}}$ and $P(A_i A_j A_k) = \frac{\binom{40}{1}}{\binom{52}{5}}$. Since the hand only contains 13 cards, it can't possible contain more than three four-of-a-kinds.

So,

$$P(\bigcup_{i=1}^{13} A_i) = \sum_{k=1}^{3} (-1)^{k+1} \binom{13}{k} \frac{\binom{52-4k}{13-4k}}{\binom{52}{5}} \approx 0.0342.$$

11. (10 pts) In a particular area, the number of traffic accidents hovers around an average of 1,050. Last year, however, the number of accidents plunged drastically to 920. Authorities suggest that the decrease is the result of new traffic safety measures that have been in effect for one year. Statistically speaking, is there cause to doubt this explanation?

Let X be the number of accidents in a year. Then X is Poisson distributed with $\lambda = 1050$. Since $\lambda \geq 25$, we may approximate X by the normal distribution $N(1050, 1050)$.

The probability that there will be 920 or fewer accidents in a year is

$$P(X \leq 1050) = P(\frac{X - 1050}{\sqrt{1050}} \leq \frac{920 - 1050}{\sqrt{1050}}) \approx P(Z \leq -4.01) = \Phi(-4.01)$$

(where Z denotes a standard normal). With a statistical calculator, we can compute that this probability is 0.00003. But, even without a calculator we know that this probability will be very small, as it's more than four standard deviations below the mean. Thus, it is unlikely that the drop in accidents is due to a chance variation and so there is no statistical reason to doubt this explanation. (Note, however, that our calculation only considered chance variation, so there may be non-statistical reasons to doubt this explanation.)

12. **(Extra Credit, 10 pts)** Let X be Poisson distributed with parameter λ. Calculate the expected value and the standard deviation of X.

We calculate:

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} kP(X=k) \\ &= \sum_{k=1}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} \\ &= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} \\ &= \lambda e^{-\lambda}e^{\lambda} \\ &= \lambda. \end{aligned}$$

Note that $\mathrm{var}(X) = E(X^2) - E(X)^2$. We calculate:

$$\begin{aligned} E(X^2) &= \sum_{k=0}^{\infty} k^2P(X=k) \\ &= \sum_{k=1}^{\infty}(k(k-1)+k)e^{-\lambda}\frac{\lambda^k}{k!} \text{ (as } k(k-1)+k = k^2-k+k = k^2) \\ &= e^{-\lambda}\sum_{k=1}^{\infty}k(k-1)\frac{\lambda^k}{k!} + e^{-\lambda}\sum_{k=1}^{\infty}k\frac{\lambda^k}{k!} \\ &= \lambda^2e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} + E(X) \\ &= \lambda^2e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} + \lambda \\ &= \lambda^2e^{-\lambda}e^{\lambda} + \lambda \\ &= \lambda^2 + \lambda, \end{aligned}$$

so that $\mathrm{var}(X) = \lambda^2 + \lambda - (\lambda)^2 = \lambda$ and $\sigma(X) = \sqrt{\mathrm{var}(X)} = \sqrt{\lambda}$.