Note: A typo was corrected in the statement of computational problem #19.

1 True/False

Examples

True or false: Answers in blue. Justification is given unless the result is a direct statement of a theorem from the book/homework or a straightforward calculation.

1. Let A be a 3×3 matrix. Then there is a pattern in A with precisely 2 inversions.
   True.

2. Let A be a 3×3 matrix. Then there is a pattern in A with precisely 3 inversions.
   True.

3. Let A be a 3×3 matrix. Then there is a pattern in A with precisely 4 inversions.
   False. We'll get the maximum number of inversions if we start in the bottom left entry and move up one and right one for each successive entry (that is, along the off-diagonal). In this case, we end up with only three inversions.

4. Let A be a 4×4 matrix. Then all patterns of A have at most 5 inversions.
   False. Again, we get the maximum by taking the pattern of all entries on the off-diagonal, which has six inversions.

5. Let A be an $n \times n$ matrix. Then $\det(A^{\mathrm{T}}) = \det(A)$.
   True.

6. Let A be an $n \times n$ matrix. Then $\det(A^{-1}) = \det(A)$.
   False. A correct statement is $\det(A^{-1}) = \frac{1}{\det(A)}$.

7. Let B be an $(n-1)\times(n-1)$ matrix and A be the $n \times n$ matrix $\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$ (where the 0 entries represent zero matrices of the appropriate size). Then $\det(A) = \det(B)$.
   True. The only nonzero patterns are the ones with the 1 selected in the upper-left, which will have the same signatures and products as the corresponding patterns in B.

8. Let A be an $n \times n$ matrix. If $\operatorname{rank}(A) \neq n$, then 0 is an eigenvalue of A.
   True. These are equivalent conditions for A not being invertible.

9. Let A be the matrix of a rotation by angle θ. Then A has no real eigenvalues.
   False. It's often true, but consider for example $\theta = 180°$.

10. If a matrix has no eigenvalues, then it has no eigenvectors.
True. (Since 7.5 isn't on the exam, this question is talking about real eigenvalues. As we saw in class, an $n \times n$ matrix over $\mathbb{C}$ always has n eigenvalues when counted according to algebraic multiplicity).

11. Let A be an $n \times n$ matrix. Let $\vec{e_1}$ be an eigenvector of A with eigenvalue 1. Then the first column of A is $\vec{e_1}$.
True, since $A\vec{e_1} = \vec{e_1}$ is the first column of A.

12. Let E_2 be an eigenspace of the matrix A. Let $\vec{v}$ be a nonzero vector in E_2. Then $A\vec{v} = 2\vec{v}$.
True by definition of eigenspace.

13. Let λ be an eigenvalue of the matrix A. Then $\dim(E_\lambda) \geq 1$.
True.

14. Let A be a 4×4 matrix and let λ be an eigenvalue of A with algebraic multiplicity 3. Then the geometric multiplicity of λ cannot be 2.
False. Counterexamples are not hard to find. For example, consider $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

15. Let A be a 4×4 matrix and let λ be an eigenvalue of A with algebraic multiplicity 3. Then the geometric multiplicity of λ cannot be 4.
True. The geometric multiplicity is always less than or equal to the algebraic multiplicity.

16. If an $n \times n$ matrix has n distinct eigenvalues, then it has an eigenbasis.
True.

17. Let A be an $n \times n$ matrix. If $\mathrm{tr}(A) = \det(A)$, then A is invertible.
False. For example, consider the zero matrix.

18. Let A be an $n \times n$ matrix. Then the eigenvalues of A are the diagonal entries of A.
False. This is true if A is a triangular matrix, but not in general.

19. Let A be a lower triangular matrix with all entries on the diagonal distinct. Then there is an eigenbasis for A.
True, since in this case we have distinct eigenvalues.

20. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to algebraic multiplicity). Then $\det(A) = \lambda_1 + \cdots + \lambda_n$.

False. This is the formula for the trace; the determinant is the product of the eigenvalues.

21. Let A be an $n \times n$ matrix with n distinct eigenvalues. If the largest of the absolute values of the eigenvalues is 0.95, then $\lim_{t\to\infty} A^t\vec{v} = \vec{0}$ for every vector $\vec{v}$ in $\mathbb{R}^n$.
True. As we've seen, this happens whenever the largest of the absolute values of the eigenvalues is < 1.

22. If A is similar to B, then $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(A) = \det(B)$.
True.

23. If $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(A) = \det(B)$, then A is similar to B.
False. See Section 7.4, #38 for a counterexample.

2 Computational

1. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 4 & 0 & 3 & 6 \\ 0 & 9 & 7 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 & 0 \end{bmatrix}$.

The only nonzero pattern P comes from selecting entries: $\begin{bmatrix} \boxed{1} & 0 & 0 & 2 & 1 \\ 0 & 4 & 0 & \boxed{3} & 6 \\ 0 & 9 & 7 & 0 & \boxed{3} \\ 0 & \boxed{-1} & 0 & 0 & 0 \\ 0 & -2 & \boxed{-3} & 0 & 0 \end{bmatrix}$.
This pattern has four inversions, so $\mathrm{sgn}(P) = 1$ and $\mathrm{prod}(P) = (1)(-1)(-3)(3)(3) = 27$, so $\det(A) = 27$.

2. Let $A = \begin{bmatrix} 1 & \boxed{2} & 3 & 4 \\ 5 & 6 & 7 & \boxed{8} \\ 9 & 10 & \boxed{11} & 12 \\ \boxed{13} & 14 & 15 & 16 \end{bmatrix}$ and let P be the pattern indicated (by the boxed entries). Find $\mathrm{sgn}(P)$ and $\mathrm{prod}(P)$ (but don't bother actually multiplying out the numbers in $\mathrm{prod}(P)$).
This pattern has four inversions, so $\mathrm{sgn}(P) = 1$ and $\mathrm{prod}(P) = (2)(8)(11)(13)$.

3. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & 3 & 6 \\ 0 & 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

   For a triangular matrix, the determinant is the product of the diagonal entries: $\det(A) = (1)(2)(-1)(2)(3) = -12$.

4. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

   Use our row-reducing strategy (or any other method you may know) to show that $\det(A) = -6$.

5. Let A be an $n \times n$ matrix. Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. Is $\vec{v}$ an eigenvector of $A^2 + 3A$? If so, what is its eigenvalue?
   Observe that $(A^2+3A)\vec{v} = A^2\vec{v} + 3A\vec{v} = A(\lambda\vec{v}) + 3(\lambda\vec{v}) = (\lambda^2+3\lambda)\vec{v}$, so $\vec{v}$ is an eigenvector of $A^2 + 3A$ with eigenvalue $\lambda^2 + 3\lambda$.

6. Find all 2×2 matrices for which $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3.

   Consider the equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solving, we see $b = 3 - a$ and $d = 3 - c$, so all matrices of the form $\begin{bmatrix} a & 3-a \\ c & 3-c \end{bmatrix}$ will have eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 3.

7. Let A be a 2×2 matrix with $\operatorname{tr}(A) = 6$ and $\det(A) = 5$. Find the eigenvalues of A.
   Since this is 2×2, we can use the information given to completely determine the characteristic polynomial (as opposed to the case in which the dimension is higher, in which case we can only find some of the coefficients from this information). Indeed, $f_A(\lambda) = \lambda^2 - 6\lambda + 5$ and factoring shows that the eigenvalues of A are 1 and 5.

8. Let A be the matrix of an orthogonal projection onto a plane V in $\mathbb{R}^3$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)
   Any vector in the plane V will map to itself and so is an eigenvector with eigenvalue 1. Any vector on the line $V^\perp$ (it's a line since we're in a 3-dimensional

space) will map to $\vec{0}$ and so is an eigenvector with eigenvalue 0. We can find an eigenbasis by picking two noncollinear vectors from V and one vector from $V^{\perp}$. See also Example 1 of Section 7.3 on page 320.

9. Let A be the matrix of a vertical shear in $\mathbb{R}^2$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)
A shear changes one coordinate of the vector (unless we are in the trivial case of a shear of strength $k = 0$, in which case $A = I_2$ and all vectors are eigenvectors) but not the other, so a vector will be a scalar multiple of itself if and only if it is unchanged by the shear. Thus, the vectors on the y-axis are eigenvectors with eigenvalue 1. Since we can find only one linearly independent vector, we will not have an eigenbasis.

   Algebraically, this comes from the fact that the matrix $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ is not diagonalizable when $k \neq 0$.

10. Let $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It can be shown that A is the matrix of a 90° counterclockwise rotation about the z-axis in $\mathbb{R}^3$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)
Any vector whose x and y coordinates are nonzero will be rotated and 90° in the (x, y)-plane and so its rotation will not be a scalar multiple of itself. Thus, the only eigenvectors are the vectors on the z-axis, with eigenvalue 1. Since we can only find one linearly independent vector, we will not have an eigenbasis.

11. Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$. It so happens that $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Let $a(t + 1) = 3a(t) + 4b(t)$ and $b(t + 1) = 4a(t) + 3b(t)$ and suppose $a(0) = 6$ and $b(0) = 2$. Find closed formulas for $a(t)$ and $b(t)$.
Let $\vec{x}(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. This problem gives you the eigenvalues and eigenvectors, so we just need to find the coordinates of our initial condition $\vec{x_0} = \vec{x}(0)$ in terms of the eigenbasis $\{\vec{v_1}, \vec{v_2}\} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ and use the formula from Section 7.1. We find (by inspection, or from Section 3.4 techniques) that $\vec{x_0} = 4\vec{v_1} + 2\vec{v_2}$, so

$\vec{x}(t) = 4(7^t)\begin{bmatrix}1\\1\end{bmatrix} + 2(-1)^t\begin{bmatrix}1\\-1\end{bmatrix}$. Thus,

$$a(t) = 4(7^t) + 2(-1)^t$$
$$b(t) = 4(7^t) - 2(-1)^t$$

12. Let $A = \begin{bmatrix}2 & 6\\-1 & 3\end{bmatrix}$. Find all real eigenvalues of A and their algebraic multiplicities.
$f_A(\lambda) = \lambda^2 - 5\lambda + 12$. From the quadratic formula, we see that A has no real eigenvalues.

13. Let $A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\0 & 2 & 1 & 3 & 9\\0 & 0 & -1 & 3 & 6\\0 & 0 & 0 & 2 & 11\\0 & 0 & 0 & 0 & 3\end{bmatrix}$. Find all real eigenvalues of A and their algebraic multiplicities.
A is triangular, so the eigenvalues are the diagonal entries: 1, 2 with algebraic multiplicity 2, -1, and 3.

14. Let A be a 2×2 matrix with eigenvalues 1 and 5. Find the characteristic polynomial of A.
The characteristic polynomial is the polynomial whose roots are the eigenvalues of A (repeated according to algebraic multiplicity). Since A is 2×2, 1 and 5 are the only eigenvalues of A and each must have algebraic multiplicity 1, so that $f_A(\lambda) = (\lambda - 1)(\lambda - 5)$.

15. Let A be a 3×3 matrix with eigenvalue 0 with algebraic multiplicity 3. Find the characteristic polynomial of A.
As above. $f_A(\lambda) = (0 - \lambda)^3 = -\lambda^3$.

16. Let A be a 2×2 matrix with $\mathrm{tr}(A) = 5$ and $\det(A) = 11$. Find the characteristic polynomial of A.
Since A is 2×2, this is enough information to find all of the coefficients of $f_A(\lambda)$. See Theorem 7.2.5 on page 311. $f_A(\lambda) = \lambda^2 - 5\lambda + 11$.

17. Let $A = \begin{bmatrix}1 & k\\k & 2\end{bmatrix}$. Find all scalars k so that 1 is an eigenvalue of A.
The characteristic polynomial is $f_A(\lambda) = \lambda^2 - 3\lambda + 2 - k^2$. 1 is an eigenvalue if and only if $f_A(1) = 0$, or in other words, if $1 - 3 + 2 - k^2 = 0$, so $k = 0$.

18. Let $A = \begin{bmatrix} 1 & k \\ k & 2 \end{bmatrix}$. Find all scalars k so that 2 is an eigenvalue of A.
Proceed as above. Again, it so happens that 2 is an eigenvalue of A if and only if $k = 0$.

19. Let $A = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$. It so happens that the eigenvalues of A are 2 and 3. Find bases for the eigenspaces E_2 and E_3. Find all real eigenvectors of A and find an eigenbasis for A if possible. (If not, explain why not.)
Note: The problem was incorrect as stated, since 2 and 3 were not eigenvalues of the given matrix. I have adjusted the matrix slightly to make it doable.
$E_2 = \ker\left(\begin{bmatrix} 0 & 6 \\ 0 & 1 \end{bmatrix}\right) = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ and $E_3 = \ker\left(\begin{bmatrix} -1 & 6 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\{\begin{bmatrix} 6 \\ 1 \end{bmatrix}\}$, so a basis of E_2 is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ and a basis of E_3 is $\{\begin{bmatrix} 6 \\ 1 \end{bmatrix}\}$. The eigenvectors of A are $c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $d\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ where c and d are nonzero scalars. An eigenbasis of A is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}\}$.

20. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It so happens that the eigenvalues of A are 1 (with algebraic multiplicity 2) and 0. Find the eigenspaces of A, all real eigenvectors of A, and find an eigenbasis for A if possible. (If not, explain why not.)
$E_1 = \ker\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right) = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$ and $E_0 = \ker(A) = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\}$. The eigenvectors of A are $c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where not both of $c \neq 0$ and $d\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ where $d \neq 0$. Since the geometric multiplicity of the eigenvalue 1 is 1, which is less that the algebraic multiplicity (i.e., 2), there is no eigenbasis for A.

21. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It so happens that the only eigenvalue of A is 1 (with algebraic multiplicity 3). Find the eigenspaces of A, all real eigenvectors of A, and find an eigenbasis for A if possible. (If not, explain why not.)

$E_1 = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$. The real eigenvectors of A are $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where not both of c_1 and c_2 are zero (geometrically, this is the xz-plane minus the origin). There is not an eigenbasis for A, since the geometric multiplicity of the eigenvalue 1 (i.e., 2) is strictly less than the algebraic multiplicity of the eigenvalue 1 (i.e., 3).

22. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ if possible (that is, find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.) If it's not possible, explain why not.

    First, find an eigenbasis. For example, $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\}$ (corresponding respectively to the eigenvalues 1, 2, and 3). Then, let $S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, so that $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

23. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ if possible (that is, find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.) If it's not possible, explain why not.

    This is not possible. The geometric multiplicity of 1 is 2, while its algebraic multiplicity is 3, so there is no eigenbasis.

24. For which constants k is the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$ diagonalizable?

    The eigenvalues are 1, 1, and k.

    If $k \neq 1$, then 1 has algebraic multiplicity 2 and k has algebraic multiplicity 1.

We find that an eigenbasis is $\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\}$, and so A is diagonalizable.

If $k = 1$, then 1 has algebraic multiplicity 3, but we calculate that is has geometric multiplicity of only 2 and so there is no eigenbasis for A, and A is not diagonalizable.

25. Let $A = \begin{bmatrix}1 & 4\\0 & 2\end{bmatrix}$. Find formula for the entries of A^t where t is a positive integer.

    Also, find the vector $A^t\begin{bmatrix}4\\1\end{bmatrix}$.

    Diagonalize A and use the formula $A^t = SD^tS^{-1}$ to compute that $A^t = \begin{bmatrix}1 & 4(2^k)-4\\0 & 2^k\end{bmatrix}$.

    Then, $A^t\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}4(2^k)\\2^k\end{bmatrix}$.

26. Let A and B be 2×2 matrices with $\det(A) = \det(B) = -1$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. Is A necessarily similar to B? (Explain why it is or give a counter-example to show that it isn't.)
    Yes. The characteristic polynomials are $f_A(\lambda) = f_B(\lambda) = \lambda^2 - 1$, so the eigenvalues of A are 1 and -1. As we have 2 distinct eigenvalues, A and B are both diagonalizable and similar to $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$ and so similar to each other by symmetry and transitivity.

27. Let A and B be 2×2 matrices with $\det(A) = \det(B) = 1$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = -2$. Is A necessarily similar to B? (Explain why it is or give a counter-example to show that it isn't.)
    No. In this case, the characteristic polynomials are both $\lambda^2 + 2\lambda + 1$, so that the only eigenvalue of each of A and B is -1 (with algebraic multiplicity 2). We can show that they need not be similar by finding examples in which the geometric multiplicity of the eigenvalue differs. It's easiest to start with triangular matrices, as then we can easily ensure that the eigenvalues are what they should be (as well as the conditions on the determinant and the trace). Doing this, we find that $A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, B = \begin{bmatrix}-1 & 1\\0 & -1\end{bmatrix}$ is a counterexample (and in fact, $A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}, B = \begin{bmatrix}-1 & b\\0 & -1\end{bmatrix}$ is a counterexample for any $b \neq 0$).