The exam will cover Sections 6.1-6.2 and 7.1-7.4:

True/False	30%
Definitions	10%
Computational	60%

Skip "Minors and Laplace Expansion" in Section 6.2 and p. 304 (trajectories and phase portraits) in Section 7.1. Note that there will not be a proof question on this exam; otherwise, the look and feel should be similar to that of the previous exams. While we've discussed vector spaces other than $\mathbb{R}^n$ briefly, you won't need to use them on the exam. The exam will **not** be cumulative, but some problems may involve calculations and/or terminology from earlier sections (for example, you need to be able to find a basis for the kernel of a matrix in order to find an eigenspace).

1 True/False

Examples

True or false:

1. Let A be a 3×3 matrix. Then there is a pattern in A with precisely 2 inversions.
2. Let A be a 3×3 matrix. Then there is a pattern in A with precisely 3 inversions.
3. Let A be a 3×3 matrix. Then there is a pattern in A with precisely 4 inversions.
4. Let A be a 4×4 matrix. Then all patterns of A have at most 5 inversions.
5. Let A be an $n\times n$ matrix. Then $\det(A^{\mathrm{T}}) = \det(A)$.
6. Let A be an $n\times n$ matrix. Then $\det(A^{-1}) = \det(A)$.
7. Let B be an $(n-1)\times(n-1)$ matrix and A be the $n\times n$ matrix $\begin{bmatrix}1 & 0\\ 0 & B\end{bmatrix}$ (where the 0 entries represent zero matrices of the appropriate size). Then $\det(A) = \det(B)$.
8. Let A be an $n\times n$ matrix. If $\operatorname{rank}(A)\neq n$, then 0 is an eigenvalue of A.
9. Let A be the matrix of a rotation by angle θ. Then A has no real eigenvalues.
10. If a matrix has no eigenvalues, then it has no eigenvectors.
11. Let A be an $n\times n$ matrix. Let $\vec{e_1}$ be an eigenvector of A with eigenvalue 1. Then the first column of A is $\vec{e_1}$.

12. Let E_2 be an eigenspace of the matrix A. Let $\vec{v}$ be a nonzero vector in E_2. Then $A\vec{v} = 2\vec{v}$.

13. Let λ be an eigenvalue of the matrix A. Then $\dim(E_\lambda) \geq 1$.

14. Let A be a 4×4 matrix and let λ be an eigenvalue of A with algebraic multiplicity 3. Then the geometric multiplicity of λ cannot be 2.

15. Let A be a 4×4 matrix and let λ be an eigenvalue of A with algebraic multiplicity 3. Then the geometric multiplicity of λ cannot be 4.

16. If an $n \times n$ matrix has n distinct eigenvalues, then it has an eigenbasis.

17. Let A be an $n \times n$ matrix. If $\operatorname{tr}(A) = \det(A)$, then A is invertible.

18. Let A be an $n \times n$ matrix. Then the eigenvalues of A are the diagonal entries of A.

19. Let A be a lower triangular matrix with all entries on the diagonal distinct. Then there is an eigenbasis for A.

20. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to algebraic multiplicity). Then $\det(A) = \lambda_1 + \cdots + \lambda_n$.

21. Let A be an $n \times n$ matrix with n distinct eigenvalues. If the largest of the absolute values of the eigenvectors is 0.95, then $\lim_{t\to\infty} A^t\vec{v} = \vec{0}$ for every vector $\vec{v}$ in $\mathbb{R}^n$.

22. If A is similar to B, then $\operatorname{tr}(A) = \operatorname{tr}(B)$ and $\det(A) = \det(B)$.

23. If $\operatorname{tr}(A) = \operatorname{tr}(B)$ and $\det(A) = \det(B)$, then A is similar to B.

24. The chapter reviews

2 Definitions

Know all things labeled as definitions.

3 Computational

These will focus on applying the various techniques we've seen to find numerical answers to problems. Some of the topics build on each other, but I'll split these into multiple problems instead (so that you won't miss latter parts due to an error in earlier parts) unless the earlier parts are really easy (for example, because the matrix is triangular). For example, instead of asking you to find eigenvalues and eigenvectors in one problem, I might instead have one problem on eigenvalues and then a different problem in which I tell you the eigenvalues and ask you to compute the eigenvectors. Note that these also require some background from earlier in the book. In particular, you'll want to be able to find the kernel of a matrix (preferably by inspection) and the coordinates of a vector with respect to a basis. See chapter 3.

1. Use the definition of the determinant; e.g., to calculate the determinant of a matrix with a lot of zeroes. (Sec. 6.1, #37, 39-42, 53, 56)
2. Calculate the signature and/or product of a pattern. (done as one step in the process of doing the above)
3. Calculate the determinant of a matrix (using memorized formulas in the 2×2 and 3×3 cases, and row-reduction for larger matrices or special methods for, say, triangular matrices). (Sec. 6.2, #1-16)
4. Basic calculations with eigenvectors/eigenvalues of matrices (Sec. 7.1, #1-6, 8-14, Sec. 7.3, #21-22)
5. Find eigenvectors and eigenvalues geometrically, and an eigenbasis if possible (Sec. 7.1, #15-22)
6. Solve dynamical systems (Sec. 7.1, #49-51)
7. Find all real eigenvalues of a matrix and their algebraic multiplicities (Sec. 7.2, #1-13, 15-18)
8. Find the characteristic polynomial of a matrix (either done as a step in the above, or use det and tr in 2×2 case) (in addition to above, Sec. 7.2, #22-23, 38, 45)
9. Find the real eigenvectors/eigenspaces associated to an eigenvalue λ and their geometric multiplicities; find an eigenbasis if possible (Sec. 7.3, #1-20)
10. Determine whether a matrix A is diagonalizable. If it is, find an invertible matrix S and diagonal matrix D such that $D = S^{-1}AS$. (Sec. 7.4, #1-30)
11. Use the above to make calculations involving powers of a matrix (Sec. 7.4, #31-34)
12. Use criteria from Section 7.4 to determine if matrices are similar (Sec. 7.4, #35-38)

As always, checking your work is a good idea. For example, if you've found an eigenvector $\vec{v}$ with eigenvalue λ for a matrix A, verify that $A\vec{v} = \lambda\vec{v}$. Also, checking the conditions on the determinant and trace will help you ensure that your characteristic polynomials and/or eigenvalues are correct.

Examples

You should be able to create your own examples easily by making up your own matrices/etc. (or use the exercises in the book above), but a few follow nonetheless.

1. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 4 & 0 & 3 & 6 \\ 0 & 9 & 7 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 & 0 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 1 & \boxed{2} & 3 & 4 \\ 5 & 6 & 7 & \boxed{8} \\ 9 & 10 & \boxed{11} & 12 \\ \boxed{13} & 14 & 15 & 16 \end{bmatrix}$ and let P be the pattern indicated (by the boxed entries). Find $\operatorname{sgn}(P)$ and $\operatorname{prod}(P)$ (but don't bother actually multiplying out the numbers in $\operatorname{prod}(P)$).

3. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & 3 & 6 \\ 0 & 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

4. Find $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

5. Let A be an $n \times n$ matrix. Let $\vec{v}$ be an eigenvector of A with eigenvalue λ. Is $\vec{v}$ an eigenvector of $A^2 + 3A$? If so, what is its eigenvalue?

6. Find all 2×2 matrices for which $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3.

7. Let A be a 2×2 matrix with $\operatorname{tr}(A) = 6$ and $\det(A) = 5$. Find the eigenvalues of A.

8. Let A be the matrix of an orthogonal projection onto a plane V in $\mathbb{R}^3$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)

9. Let A be the matrix of a vertical shear in $\mathbb{R}^2$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)

10. Let $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It can be shown that A is the matrix of a $90°$ counterclockwise rotation about the z-axis in $\mathbb{R}^3$. Arguing geometrically, find all real eigenvectors and eigenvalues of A and find an eigenbasis if possible. (If not possible, explain why not.)

11. Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$. It so happens that $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Let $a(t+1) = 3a(t) + 4b(t)$ and $b(t+1) = 4a(t) + 3b(t)$ and suppose $a(0) = 6$ and $b(0) = 2$. Find closed formulas for $a(t)$ and $b(t)$.

12. Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. Find all real eigenvalues of A and their algebraic multiplicities.

13. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & 3 & 9 \\ 0 & 0 & -1 & 3 & 6 \\ 0 & 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. Find all real eigenvalues of A and their algebraic multiplicities.

14. Let A be a 2×2 matrix with eigenvalues 1 and 5. Find the characteristic polynomial of A.

15. Let A be a 3×3 matrix with eigenvalue 0 with algebraic multiplicity 3. Find the characteristic polynomial of A.

16. Let A be a 2×2 matrix with $\operatorname{tr}(A) = 5$ and $\det(A) = 11$. Find the characteristic polynomial of A.

17. Let $A = \begin{bmatrix} 1 & k \\ k & 2 \end{bmatrix}$. Find all scalars k so that 1 is an eigenvalue of A.

18. Let $A = \begin{bmatrix} 1 & k \\ k & 2 \end{bmatrix}$. Find all scalars k so that 2 is an eigenvalue of A.

19. Let $A = \begin{bmatrix} 2 & 6 \\ -1 & 3 \end{bmatrix}$. It so happens that the eigenvalues of A are 2 and 3. Find bases for the eigenspaces E_2 and E_3. Find all real eigenvectors of A and find an eigenbasis for A if possible. (If not, explain why not.)

20. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It so happens that the eigenvalues of A are 1 (with algebraic multiplicity 2) and 0. Find the eigenspaces of A, all real eigenvectors of A, and find an eigenbasis for A if possible. (If not, explain why not.)

21. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It so happens that the only eigenvalue of A is 1 (with algebraic multiplicity 3). Find the eigenspaces of A, all real eigenvectors of A, and find an eigenbasis for A if possible. (If not, explain why not.)

(The problems below require techniques from Section 7.4)

22. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ if possible (that is, find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.) If it's not possible, explain why not.

23. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ if possible (that is, find an invertible matrix S and a diagonal matrix D such that $D = S^{-1}AS$.) If it's not possible, explain why not.

24. For which constants k is the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$ diagonalizable?

25. Let $A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$. Find formula for the entries of A^t where t is a positive integer. Also, find the vector $A^t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

26. Let A and B be 2×2 matrices with $\det(A) = \det(B) = -1$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. Is A necessarily similar to B? (Explain why it is or give a counter-example to show that it isn't.)

27. Let A and B be 2×2 matrices with $\det(A) = \det(B) = 1$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = -2$. Is A necessarily similar to B? (Explain why it is or give a counter-example to show that it isn't.)