Note: Typos were corrected in the statements of True/False #2 and Computational #1, #6.

1 True/False

Examples

True or false: Answers in blue. Justification is given unless the result is a direct statement of a theorem from the book/homework.

1. Let $V = \{f \text{ in } C^\infty \mid f'(x) \neq 0 \text{ for all } x\}$. Then V is a subspace of C^∞.
   False; e.g., V does not contain the zero vector $f(x) = 0$.

2. Let $T(f) = f(0)$ be a linear transformation from $\mathcal{P}_3$ to $\mathbb{R}$. Then T is an isomorphism.
   False; $\dim(\mathcal{P}_3) = 4$, $\dim(\mathbb{R}) = 1$, so the spaces are not isomorphic.

3. $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^n$.
   False; $\dim(\mathcal{P}_n) = n + 1$, $\dim(\mathbb{R}^n) = n$.

4. $\mathcal{P}_{11}$ is isomorphic to $\mathbb{R}^{6\times 2}$.
   True; one isomorphism is to arrange the values $f(1), \ldots, f(12)$ as the twelve entries of the matrix. Another is to arrange the coefficients of the terms of f as the entries of the matrix.

5. There is a basis of $\mathbb{R}^{2\times 2}$ consisting of four diagonal matrices.
   False; any linear combination of diagonal matrices will be a diagonal matrix, so it's impossible to span $\mathbb{R}^{2\times 2}$ with diagonal matrices.

6. Let A and B be $n \times n$ matrices. If A is similar to B, then $A = B$.
   False; there exists an invertible matrix S such that $A = S^{-1}BS$, but it's not necessarily true that $A = B$. We've seen examples of matrices which are similar but not equal in class.

7. Let A and B be $n \times n$ matrices. If A is similar to B, then B is similar to A.
   True.

8. Let V be a finite dimensional subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $T(f) = f'$ from V to V is a linear transformation. Then T is not an isomorphism.
   False; it depends on the space V. See for example Section 4.3, #48.

9. Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional, then W is finite dimensional.
   False; e.g., $T(f) = f$ from $\mathcal{F}(\mathbb{R}, \mathbb{R})$ to $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

10. Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\text{im}(T)$ is finite dimensional, then V is finite dimensional.
True. $\dim(V) = \text{rank}(T) + \text{nullity}(T)$.

11. Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\text{im}(T)$ is finite dimensional, then W is finite dimensional.
False; e.g., $T(f) = f$ from $\mathcal{P}_1$ to $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

12. Let T be a linear transformation from a vector space V to a vector space W. If W is finite dimensional, then $\dim(W) = \text{rank}(T) + \ker(T)$.
False; this statement doesn't even make sense: you can't add a number ($\text{rank}(T)$) to a vector space ($\ker(T)$). Modifying this, the similar statement $\dim(V) = \text{rank}(T) + \text{nullity}(T)$ is true.

13. Let $\mathcal{U} = \{\vec{u_1}, \ldots, \vec{u_n}\}$ and $\mathcal{B} = \{\vec{b_1}, \ldots, \vec{b_n}\}$ be two bases of a vector space V. Then the change of basis matrix S from $\mathcal{U}$ to $\mathcal{B}$ is given by
$$S = \begin{bmatrix} [\vec{b_1}]_{\mathcal{U}} & \ldots & [\vec{b_n}]_{\mathcal{U}} \end{bmatrix}.$$
False; this is the change of basis matrix from $\mathcal{B}$ to $\mathcal{U}$.

14. Let $\mathcal{B}$ and $\mathcal{U}$ be two bases of a vector space V. If S is the change of basis matrix from $\mathcal{B}$ to $\mathcal{U}$, then S^{-1} is the change of basis matrix from $\mathcal{U}$ to $\mathcal{B}$.
True.

15. The matrix of a linear transformation from V to V is uniquely determined.
False; it changes depending on what basis you pick. However, the resulting matrices will be similar.

16. Let A be an $n \times n$ matrix. If $A^{\mathrm{T}} = A^{-1}$, then the columns of A form an orthonormal basis of $\mathbb{R}^n$.
True.

17. If A is an orthogonal $n \times n$ matrix, then the least-squares solution to $A\vec{x} = \vec{b}$ is unique and $\vec{x^\star} = A^{\mathrm{T}}\vec{b}$.
True; A is invertible, so the system is consistent. Thus, the least-squares solutions are the exact solutions, of which there is only the stated solution since $A^{-1} = A^{\mathrm{T}}$.

18. Let A be an $n \times m$ matrix. If the least-squares solution to $A\vec{x} = \vec{b}$ is unique, then $\ker(A) = \{0\}$.

True. The least-squares solutions are the vectors $\vec{x^\star}$ which satisfy $A\vec{x^\star} = \vec{y^\star}$ and we've seen previously that you can get all solutions to this equation by picking one solution and adding a vector in the kernel. If there are no other solutions, it must be that $\ker(A) = \{0\}$.

19. If $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis for $\mathbb{R}^n$, then for $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} = (\vec{v_1} \cdot \vec{x})\vec{v_1} + \cdots + (\vec{v_n} \cdot \vec{x})\vec{v_n}$.
False. This is only true for an *orthonormal* basis.

20. If A is a symmetric $n \times n$ matrix, then $A^2 = I_n$.
False; e.g., $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. However, it is true if A is both symmetric and orthogonal (since then $A = A^{\mathrm{T}} = A^{-1}$).

21. Let f and g be nonorthogonal vectors in the vector space V. Then

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

False; see Theorem 5.1.9 (the Pythagorean Theorem).

22. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Then $|\vec{x} \cdot \vec{y}| = \|\vec{x}\|\|\vec{y}\|$ if and only if $\vec{x}$ and $\vec{y}$ are parallel.
True; see Theorem 5.1.11 (the Cauchy-Schwarz inequality).

23. Let T be a linear transformation from a vector space V to $\mathbb{R}^n$. Then

$$\langle f, g \rangle = T(f) \cdot T(g)$$

is an inner product on V.
False; see #17 in Section 5.5.

24. Let $\langle f, g \rangle$ be an inner product on a vector space V. If $\langle f, g \rangle = 0$, then either $f = 0$ or $g = 0$.
False; this is true for any orthogonal vectors.

2 Proofs

1. (subspace) Let T be a transformation from V to W. Prove that $\ker(T)$ is a subspace of V.

   (a) $T(0) = T(0 + 0) = T(0) + T(0)$, so $T(0) = 0$. Thus, 0 is in $\ker(T)$.

(b) Let f, g be in $\ker(T)$. Then $T(f+g) = T(f) + T(g) = 0 + 0$, so $f+g$ is in $\ker(T)$.

(c) Let f be in $\ker(T)$ and k a scalar. Then $T(kf) = kT(f) = k0 = 0$, so kf is in $\ker(T)$.

Thus, $\ker(T)$ is a subspace of V.

2. (linear) Let S and T be linear transformations and k a scalar. Show that $S+T$ and kT are linear transformations.
Let $R(f) = S(f) + T(f)$. Then $R(f+g) = S(f+g) + T(f+g) = S(f) + S(g) + T(f) + T(g) = S(f) + T(f) + S(g) + T(g) = R(f) + R(g)$ and $R(kf) = S(kf) + T(kf) = kS(f) + k(T(f) = kR(f)$, so R is a linear transformation.

Let $Q(f) = kT(f)$. Then $Q(f+g) = kT(f+g) = k(T(f)+T(g)) = Q(f)+Q(g)$ and $Q(cf) = kT(cf) = kcT(f) = cQ(f)$, so Q is a linear transformation.

3. (isomorphism) Let $T(f) = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(n) \end{bmatrix}$ from $\mathcal{P}_n$ to $\mathbb{R}^{n+1}$. Show that T is an isomorphism.
We can use any of the parts of Thereom 4.2.4. For example, let's show that $\ker(T) = \{0\}$ (which is enough, since all of the vector spaces involved are finite dimensional). $\ker(T)$ is the set of vectors f such that $f(0) = f(1) = \cdots = f(n) = 0$. Since a nonzero polynomial of degree at most n has at most n roots and f has $n+1$ roots, it must be that $f(x) = 0$, so that $\ker(T) = \{0\}$.

Alternatively, you can also show that T is an isomorphism by showing that $\text{im}(T) = \mathbb{R}^{n+1}$ by finding polynomials which map to the basis vectors $\vec{e_1}, \ldots, \vec{e_{n+1}}$.

5. (orthogonal vectors) Let T be an orthogonal transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ be orthogonal. Then $T(\vec{v})$ and $T(\vec{w})$ are orthogonal.
See Theorem 5.3.2 (p. 211).

6. (orthogonal transformation/matrix) Let A and B be orthogonal $n \times n$ matrices. Show that AB, A^{-1}, A^{T}, and $A^{\mathrm{T}}B$ are orthogonal.
See Theorem 5.3.4 (p. 213) for the first two. Use the fact that $A^{\mathrm{T}} = A^{-1}$ for all orthogonal matrices for the third. Combine the first and the third for the fourth.

8. (inner product) Let $\langle f, g\rangle$ and (f, g) be two inner products on V. Let $k > 0$ be a scalar. Show that $\langle f, g\rangle + (f, g)$ and $k\langle f, g\rangle$ are inner products on V.
Let $[f, g] = \langle f, g\rangle + (f, g)$. Then:

   (a) $[f, g] = \langle f, g\rangle + (f, g) = \langle g, f\rangle + (g, f) = [g, f]$.

   (b) $[f + h, g] = \langle f + h, g\rangle + (f + h, g) = \langle f, g\rangle + \langle h, g\rangle + (f, g) + (h, g) == \langle f, g\rangle + (f, g) + \langle h, g\rangle + (h, g) = [f, g] + [h, g]$.

   (c) $[kf, g] = \langle kf, g\rangle + (kf, g) = k\langle f, g\rangle + k(f, g) = k[f, g]$.

   (d) Let $f \neq 0$. Then $[f, f] = \langle f, f\rangle + (f, f) > 0$ since both $\langle f, f\rangle > 0$ and $(f, f) > 0$.

   Thus, $[,]$ is an inner product on V.

   Now, let $((f, g)) = k\langle f, g\rangle$. Then:

   (a) $((f, g)) = k\langle f, g\rangle = k\langle g, f\rangle = ((g, f))$.

   (b) $((f + h, g)) = k\langle f + h, g\rangle = k(\langle f, g\rangle + \langle h, g\rangle) = k\langle f, g\rangle + k\langle h, g\rangle = ((f, g)) + ((h, g))$.

   (c) $((cf, g)) = k\langle cf, g\rangle = kc\langle f, g\rangle = ck\langle f, g\rangle = c((f, g))$.

   (d) Let $f \neq 0$. Then $((f, f)) = k\langle f, f\rangle > 0$ since both $\langle f, f\rangle > 0$ and $k > 0$.

   Thus, $((,))$ is an inner product on V.

3 Computational

1. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \}$. Let $\vec{x} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

   Either by inspection or by solving $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$, we see that $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

2. Let $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \}$. Find the $\mathcal{B}$-matrix of T.

   $B = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$.

3. Is matrix $\begin{bmatrix} 2 & -7 \\ 7 & 2 \end{bmatrix}$ similar to $\begin{bmatrix} 2 & 7 \\ -7 & 2 \end{bmatrix}$? (Justify your answer.)

   Yes. Let $S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ (for example), and then $A = S^{-1}BS$.

4. Let W be the vector space of all symmetric 4×4 matrices. Find a basis for W.
   Let E_{ij} be the matrix with a 1 in position ij and 0's elsewhere. Then a basis is the matrices $E_{11}, E_{22}, E_{33}, E_{44}$ together with the six matrices $E_{ij} + E_{ji}$ where $i < j$.

5. Let W be the subspace of $\mathcal{P}_3$ of all functions f such that $f(0) = f(1)$. Find a basis for W.
   Write $f(x) = a + bx + cx^2 + dx^3$. Then we have $a = a + b + c + d$, so $b + c + d = 0$, or $d = -b - c$. Thus, an arbitrary element is $f(x) = a + bx + cx^2 + (-b - c)x^3 = a(1) + b(x - x^3) + c(x^2 - x^3)$. After verifying that these are linearly independent, we see that $\{1, x - x^3, x^2 - x^3\}$ is a basis for W.

6. Let $T(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} A$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Find $\operatorname{im}(T)$ and $\ker(T)$.

   (I originally left an A out of the definition of T. As written before, T wasn't a linear transformation. The answer below is for the corrected version stated here.)

   Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $T(A) = \begin{bmatrix} a + 2c & b + 2d \\ 0 & 0 \end{bmatrix} = (a + 2c)E_{11} + (b + 2d)E_{12}$, so $\operatorname{im}(T) = \operatorname{span}\{E_{11}, E_{12}\}$.

   If $T(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $a = -2c$ and $b = -2d$, so $A = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix} = c\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$, so $\ker(T) = \operatorname{span}\{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}\}$.

7. Let $T(f) = f + 3f''$ from $\mathcal{P}_2$ to $\mathcal{P}_2$. Find $\operatorname{im}(T)$ and $\ker(T)$.
   Write $f(t) = a + bt + ct^2$. Then $T(f(t)) = a + bt + ct^2 + 3(2c) = a(1) + b(t) + c(6 + t^2)$, so $\operatorname{im}(T) = \operatorname{span}\{1, t, 6 + t^2\}$ (and it's also equal to $\mathcal{P}_2$).

   If $T(f) = 0$, then $a + 6c = 0$, $b = 0$, and $c = 0$. Solving, we see $a = b = c = 0$, so $\ker(T) = \{0\}$. (Incidentally, this shows that T is an isomorphism.)

8. Let $T(A) = A + A^{\mathrm{T}}$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Let $\mathcal{U} = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ be a basis of $\mathbb{R}^{2\times 2}$. Find the $\mathcal{U}$-matrix of T.

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

9. Let $\mathcal{U} = \{1, x, x^2\}$ and $\mathcal{B} = \{1, 1+x, (1+x)^2\}$ be bases for $\mathcal{P}_2$. Find the change of basis matrices from $\mathcal{B}$ to $\mathcal{U}$ and from $\mathcal{U}$ to $\mathcal{B}$.

   From $\mathcal{B}$ to $\mathcal{U}$: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ From $\mathcal{U}$ to $\mathcal{B}$: $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

10. Use the Gram-Schmidt process to find an orthonormal basis of $V = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\}$ and in the process find the QR factorization of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$.

    Write $\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

    Then $\vec{u_1} = \frac{1}{\sqrt{6}}\vec{v_1} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ and $r_{11} = \sqrt{6}$.

    Thus, $\vec{v_2}^{\perp} = \vec{v_2} - (\vec{u_1} \cdot \vec{v_2})\vec{u_1} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $r_{12} = \sqrt{6}$. (Before going on, you should verify that $\vec{u_1}$ and $\vec{v_2}^{\perp}$ really are orthogonal: if they aren't, you made a computational error and if you don't fix it now, everything that follows will be wrong too.)

Thus, $\vec{u_2} = \frac{1}{\sqrt{3}}\vec{v_2}^{\perp} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}$ and $r_{22} = \sqrt{3}$.

Now, $\vec{v_3}^{\perp} = \vec{v_3} - (\vec{u_1}\cdot\vec{v_3})\vec{u_1} - (\vec{u_2}\cdot\vec{v_3})\vec{u_2} = \frac{1}{2}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}$, and $r_{13} = \frac{3}{\sqrt{6}}, r_{23} = 0$. (Before going on, verify that $\vec{v_3}^{\perp}$ is orthogonal to both $\vec{u_1}$ and $\vec{u_2}$.)

Finally, $\vec{u_3} = \frac{1}{\sqrt{3/2}}\vec{v_3}^{\perp} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}$ and $r_{33} = \sqrt{3/2}$.

Thus, our orthonormal basis for V is $\{\dfrac{1}{\sqrt{6}}\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \dfrac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}, \dfrac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}\}$, and the QR factorization is

$$Q = \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & 1/\sqrt{3} & -2/\sqrt{6}\\ 1/\sqrt{6} & -1/\sqrt{3} & -1/\sqrt{6}\\ 2/\sqrt{6} & 0 & 0\end{bmatrix}, R = \begin{bmatrix}\sqrt{6} & \sqrt{6} & 3/\sqrt{6}\\ 0 & \sqrt{3} & 0\\ 0 & 0 & \sqrt{3/2}\end{bmatrix}.$$

(Theoretically, you can check your work one final time by multiplying QR and seeing if you end up with the matrix you started with, but since the numbers usually get messy and since most errors will be caught by checking orthogonality, this usually isn't worth it.)

11. Find the matrix of the orthogonal projection from $\mathbb{R}^3$ onto the subspace $W = \text{span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\}$.

We'd like to use Theorem 5.3.10, but **first we need an orthonormal basis**. Use Gram-Schmidt (it's really easy on these particular vectors) to find the orthonormal basis $\dfrac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$.

Thus, $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \end{bmatrix}$ and $QQ^{\mathrm{T}} = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

12. Find the least-squares solution to the inconsistent system

$$\begin{array}{rcrcr} x & + & 4y & = & -2 \\ x & + & 2y & = & 6 \\ 2x & + & 3y & = & 1. \end{array}$$

$x = 3, y = -1.$

13. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(1,1)$, $(4,2)$, $(8,4)$, $(11,5)$ using least-squares.
$f(t) = \frac{15}{29} + \frac{12}{29}t.$

14. Let $\langle A, B\rangle = \mathrm{trace}(A^{\mathrm{T}}B)$ be an inner product on $\mathbb{R}^{2\times 2}$. Pick a few matrices and compute $\langle A, B\rangle$, $\|A\|$, and $\mathrm{dist}(A, B)$.
For example, let $A = I_2$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\langle A, B\rangle = 1$, $\|A\| = 1$, $\|B\| = \sqrt{2}$, and $\mathrm{dist}(A, B) = \sqrt{3}$.