The exam will cover Section 3.4-Section 5.5 and be composed approximately as follows:

True/False	30%
Definitions	10%
Proofs	10%
Computational	50%

The exam does not cover the section "Correlation" in 5.1 or "Fourier Analysis" in 5.5. The basic look-and-feel should be similar to that of the first exam.

1 True/False

Examples
True or false:

1. Let $V = \{f \text{ in } C^\infty \mid f'(x) \neq 0 \text{ for all } x\}$. Then V is a subspace of C^∞.
2. Let $T(f) = f(0)$ be a linear transformation from $\mathcal{P}_3$ to $\mathbb{R}$. Then T is an isomorphism.
3. $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^n$.
4. $\mathcal{P}_{11}$ is isomorphic to $\mathbb{R}^{6\times 2}$.
5. There is a basis of $\mathbb{R}^{2\times 2}$ consisting of four diagonal matrices.
6. Let A and B be $n \times n$ matrices. If A is similar to B, then $A = B$.
7. Let A and B be $n \times n$ matrices. If A is similar to B, then B is similar to A.
8. Let V be a finite dimensional subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $T(f) = f'$ from V to V is a linear transformation. Then T is not an isomorphism.
9. Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional, then W is finite dimensional.
10. Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\mathrm{im}(T)$ is finite dimensional, then V is finite dimensional.
11. Let T be a linear transformation from a vector space V to a vector space W. If $\ker(T)$ is finite dimensional and $\mathrm{im}(T)$ is finite dimensional, then W is finite dimensional.

12. Let T be a linear transformation from a vector space V to a vector space W. If W is finite dimensional, then $\dim(W) = \text{rank}(T) + \ker(T)$.

13. Let $\mathcal{U} = \{\vec{u_1}, \ldots, \vec{u_n}\}$ and $\mathcal{B} = \{\vec{b_1}, \ldots, \vec{b_n}\}$ be two bases of a vector space V. Then the change of basis matrix S from $\mathcal{U}$ to $\mathcal{B}$ is given by

$$S = \begin{bmatrix} [\vec{b_1}]_{\mathcal{U}} & \ldots & [\vec{b_n}]_{\mathcal{U}} \end{bmatrix}.$$

14. Let $\mathcal{B}$ and $\mathcal{U}$ be two bases of a vector space V. If S is the change of basis matrix from $\mathcal{B}$ to $\mathcal{U}$, then S^{-1} is the change of basis matrix from $\mathcal{U}$ to $\mathcal{B}$.

15. The matrix of a linear transformation from V to V is uniquely determined.

16. Let A be an $n \times n$ matrix. If $A^{\mathrm{T}} = A^{-1}$, then the columns of A form an orthonormal basis of $\mathbb{R}^n$.

17. If A is an orthogonal $n \times n$ matrix, then the least-squares solution to $A\vec{x} = \vec{b}$ is unique and $\vec{x^\star} = A^{\mathrm{T}}\vec{b}$.

18. Let A be an $n \times m$ matrix. If the least-squares solution to $A\vec{x} = \vec{b}$ is unique, then $\ker(A) = \{0\}$.

19. If $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis for $\mathbb{R}^n$, then for $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} = (\vec{v_1} \cdot \vec{x})\vec{v_1} + \cdots + (\vec{v_n} \cdot \vec{x})\vec{v_n}$.

20. If A is a symmetric $n \times n$ matrix, then $A^2 = I_n$.

21. Let f and g be nonorthogonal vectors in the vector space V. Then

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2.$$

22. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Then $|\vec{x} \cdot \vec{y}| = \|\vec{x}\|\|\vec{y}\|$ if and only if $\vec{x}$ and $\vec{y}$ are parallel.

23. Let T be a linear transformation from a vector space V to $\mathbb{R}^n$. Then

$$\langle f, g \rangle = T(f) \cdot T(g)$$

is an inner product on V.

24. Let $\langle f, g \rangle$ be an inner product on a vector space V. If $\langle f, g \rangle = 0$, then either $f = 0$ or $g = 0$.

25. The chapter reviews

2 Definitions

Know all things labeled as definitions as well as the *trace* of a matrix. Also, know what the following are: $\mathcal{F}(\mathbb{R},\mathbb{R})$, $\mathcal{P}_n$, $\mathcal{P}$, $C[a,b]$, C^∞, $\mathbb{C}$, $\mathbb{R}^{n\times m}$. (For those that are finite dimensional, knowing a basis is also a good idea.)

3 Proofs

Note that this may be either abstract or concrete. That is, you might be asked to work with abstract concepts (as on the first exam), or with particular examples (like the problems #1-50 in 4.2). Know how to show (i.e., prove) the following:

1. A set W is a subspace of a vector space V.
2. that a transformation is linear (by showing $T(f+g) = T(f)+T(g)$ and $T(kf) = kT(f)$).
3. that a linear transformation is an isomorphism (Thm. 4.2.4, or the figure on the following page).
4. that a set of vectors is orthogonal (directly, or using the Pythagorean Theorem).
5. that a linear transformation/matrix is orthogonal (Summary 5.3.8).
6. that $\langle f, g\rangle$ is an inner product.

You may use any theorems we've seen in your proof (unless you're being asked to prove the theorem itself, of course).

Examples

1. (subspace) Let T be a transformation from V to W. Prove that $\ker(T)$ is a subspace of V.
2. (linear) Let S and T be linear transformations and k a scalar. Show that $S+T$ and kT are linear transformations.
3. (isomorphism) Let $T(f) = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(n) \end{bmatrix}$ from $\mathcal{P}_n$ to $\mathbb{R}^{n+1}$. Show that T is an isomorphism.

4. (linear, isomorphism) See also Section 4.2, #1-32, #37-50

5. (orthogonal vectors) Let T be an orthogonal transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ be orthogonal. Then $T(\vec{v})$ and $T(\vec{w})$ are orthogonal.

6. (orthogonal transformation/matrix) Let A and B be orthogonal $n \times n$ matrices. Show that AB, A^{-1}, A^{T}, and $A^{\mathrm{T}}B$ are orthogonal.

7. (orthogonal transformation/matrix) See also Sec. 6.3, #1-11.

8. (inner product) Let $\langle f, g\rangle$ and (f, g) be two inner products on V. Let $k > 0$ be a scalar. Show that $\langle f, g\rangle + (f, g)$ and $k\langle f, g\rangle$ are inner products on V.

9. (inner product) See also Sec. 5.5, #3,5,7,14,15,17,19,21.

(I probably won't ask you to prove one of those specific examples, so be sure you're focusing on how the techniques are used and not on the details.)

4 Computational

These will focus on applying the various techniques we've seen to find (usually, numerical) answers to problems. For problems involving abstract vector spaces (that is, chapter 4 stuff), concentrate on subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ (especially $\mathcal{P}_n$ for small n) and $\mathbb{R}^{n\times m}$ for small n and m (although other vector spaces might be used on the exam also). In particular, you may be asked to:

1. Find the $\mathcal{B}$-coordinates of a vector in $\mathbb{R}^n$ (Sec. 3.4, #1-18)

2. Find the $\mathcal{B}$-matrix of a linear transformation (Sec. 3.4, #19-30)

3. Show that two given matrices are or are not similar (Sec. 3.4, #59-60)

4. Verify that vectors are linearly independent, find a basis for a vector space, compute its dimension (Sec. 4.1, #1-5, 16-36, Sec. 5.3, #52-55)

5. Find the image and kernel of a linear transformation; check if it's an isomorphism (Try this on Sec. 4.2, #1-32, 37-50. It might be too hard on some of them. Also, Sec. 4.3, #5-40, Sec. 5.3 #56-59)

6. Find the $\mathcal{B}$-matrix of a linear transformation. Find the change of basis matrix S from $\mathcal{B}$ to $\mathcal{U}$. (Sec. 4.3, #41-47, 48-50)

7. Various orthogonality calculation. (Sec. 5.1, #1-10, 21)

8. Gram-Schmidt and QR factorization. (Sec. 5.2, #1-28)

9. Find the matrix of an orthogonal projection. (Sec. 5.3, #39-42)

10. Find the least-squares solutions of the equation $A\vec{x} = \vec{b}$. (Sec. 5.4, #19-26, 30-34)

11. Calculations with inner products (take inner products, compute norms and distance, show vectors are orthogonal)

Note that you can check your answers to many of these. In particular, **YOU SHOULD CHECK YOUR WORK AFTER EVERY STEP WHEN DOING GRAM-SCHMIDT. OTHERWISE, YOU WILL MAKE A MISTAKE.**

Examples

You should be able to create your own examples easily by making up your own matrices/systems/etc. (or use the exercises in the book above), but a few follow nonetheless.

1. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \}$. Let $\vec{x} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

2. Let $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$. Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix} \}$. Find the $\mathcal{B}$-matrix of T.

3. Is matrix $\begin{bmatrix} 2 & -7 \\ 7 & 2 \end{bmatrix}$ similar to $\begin{bmatrix} 2 & 7 \\ -7 & 2 \end{bmatrix}$? (Justify your answer.)

4. Let W be the vector space of all symmetric 4×4 matrices. Find a basis for W.

5. Let W be the subspace of $\mathcal{P}_3$ of all functions f such that $f(0) = f(1)$. Find a basis for W.

6. Let $T(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} A$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Find $\text{im}(T)$ and $\ker(T)$.

7. Let $T(f) = f + 3f''$ from $\mathcal{P}_2$ to $\mathcal{P}_2$. Find $\text{im}(T)$ and $\ker(T)$.

8. Let $T(A) = A + A^{\mathrm{T}}$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Let $\mathcal{U} = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$ be a basis of $\mathbb{R}^{2\times 2}$. Find the $\mathcal{U}$-matrix of T.

9. Let $\mathcal{U} = \{1, x, x^2\}$ and $\mathcal{B} = \{1, 1 + x, (1 + x)^2\}$ be bases for $\mathcal{P}_2$. Find the change of basis matrices from $\mathcal{B}$ to $\mathcal{U}$ and from $\mathcal{U}$ to $\mathcal{B}$.

10. Use the Gram-Schmidt process to find an orthonormal basis of $V = \text{span}\{\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\}$ and in the process find the QR factorization of the matrix $\begin{bmatrix}1 & 2 & 1\\0 & 1 & -1\\1 & 0 & 0\\2 & 2 & 1\end{bmatrix}$.

11. Find the matrix of the orthogonal projection from $\mathbb{R}^3$ onto the subspace $W = \text{span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\}$.

12. Find the least-squares solution to the inconsistent system

$$\begin{array}{rcrcr} x & + & 4y & = & -2 \\ x & + & 2y & = & 6 \\ 2x & + & 3y & = & 1. \end{array}$$

13. Fit a linear function of the form $f(t) = c_0 + c_1 t$ to the data points $(1,1)$, $(4,2)$, $(8,4)$, $(11,5)$ using least-squares.

14. Let $\langle A, B\rangle = \text{trace}(A^{\mathrm{T}}B)$ be an inner product on $\mathbb{R}^{2\times 2}$. Pick a few matrices and compute $\langle A, B\rangle$, $\|A\|$, and $\text{dist}(A, B)$.