1 True/False

True or false: Answers in blue. Justification is given unless the result is a direct statement of a theorem from the book/homework.

1. If a system of equations has fewer equations than unknowns, then it has infinitely many solutions.
   False; it could have no solution.

2. If A in an $n \times m$ matrix, then $\text{rank}(A) \leq n$.
   True.

3. If A in an $n \times n$ matrix and $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.
   False; this is only true if $\text{rank}(A) = n$. e.g., $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4. If a square matrix has two equal columns, then it is not invertible.
   True.

5. If a square matrix has two equal rows, then it is not invertible.
   True. $\text{rref}(A)$ will have a row of zeroes, so $\text{rref}(A) \neq I_n$.

6. There exists a 2×2 matrix A such that $\text{rank}(A) = 0$.
   True; the zero matrix.

7. There exists a 2×2 matrix A such that $\text{rank}(A) = 4$.
   False; $\text{rank}(A) \leq 2$.

8. If A and B are $n \times n$ matrices, then $(AB)^2 = A^2B^2$.
   False; only true if $AB = BA$. Otherwise, $(AB)^2 = ABAB$.

9. For all $n \times n$ matrices A, B, and C, $(AB)C = A(BC)$.
   True.

10. A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2 + b^2 = 1$ must be invertible.
    True. $\det(A) = 1$. Also, clearly invertible from the geometric interpretation as a rotation.

11. There exists a 2×2 matrix A such that $A^3 = I_2$ but $A \neq I_2$.
    True. A rotation by $2\pi/3$ radians will work.

12. There exists a 2×2 matrix A such that $A^4 = I_2$ but $A^2 \neq I_2$.
    True. A rotation by $2\pi/4$ radians will work.

13. There exists a 2×2 matrix A such that $A^2 = I_2$ but $A^4 \neq I_2$.
False; if $A^2 = I_2$, then $A^4 = (A^2)^2 = (I_2)^2 = I_2$.

14. If A is a 3×4 matrix, then $A\vec{x} = \vec{0}$ has infinitely many solutions.
True; fewer equations than variables, so not a unique solution, and homogeneous, so at least one solution.

15. The solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{e_1}$ form a line in $\mathbb{R}^2$.
False; this system is inconsistent. However, the solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a line in $\mathbb{R}^2$.

16. Let A and B be $n \times n$ matrices. If $\vec{v}$ is in $\ker(B)$, then $\vec{v}$ is in $\ker(AB)$.
True; $AB\vec{v} = A\vec{0} = \vec{0}$.

17. Let A and B be $n \times n$ matrices. If $\vec{v}$ is in $\ker(B)$, then $\vec{v}$ is in $\ker(BA)$.
False; this almost never happens, so counterexamples are easy to find. e.g., let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $\vec{v} = \vec{e_2}$.

18. If $\vec{x} = \vec{v}$ and $\vec{x} = \vec{w}$ are two solutions to $A\vec{x} = \vec{b}$, then $\vec{x} = \vec{v} + \vec{w}$ is a solution too.
False; this is only true if $\vec{b} = \vec{0}$. Otherwise, $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = 2\vec{b}$.

19. If $\vec{v}$ and $\vec{w}$ are in $\operatorname{im}(A)$, then $2\vec{v} - 7\vec{w}$ is in $\operatorname{im}(A)$ too.
True; $\operatorname{im}(A)$ is a subspace of $\mathbb{R}^n$, so it's closed under linear combinations.

20. If $A\vec{v} = A\vec{w}$, then $\vec{v} - \vec{w}$ is in $\ker(A)$.
True; $A(\vec{v} - \vec{w}) = A\vec{v} - A\vec{w} = \vec{0}$.

21. Let A be an $n \times m$ matrix. Then $\operatorname{im}(A)$ is a subspace of $\mathbb{R}^n$.
True.

22. Let A be an $n \times m$ matrix. Then $\operatorname{im}(A)$ is a subspace of $\mathbb{R}^m$.
False. $\operatorname{im}(A)$ is a subspace of $\mathbb{R}^n$.

23. If $\ker(A) = \{\vec{0}\}$ for an $n \times m$ matrix A, then $n \leq m$.
False. Rather, $m \leq n$.

24. If A is an upper-triangular matrix, then A is invertible.
False. This is only true if the entries on the diagonal are all nonzero, as then $\operatorname{rref}(A) = I_n$.

25. If A is a permutation matrix, then $A\vec{e_1} = \vec{e_i}$ for some i.
True. Clear from the definition of a permutation matrix.

26. Let A be a $n \times m$ matrix. Then $\dim(\text{im}(A)) + \dim(\ker(A)) = n$.
False. By Rank-Nullity, it's equal to m.

2 Proofs

1. (vector equality) Let A be an $n \times m$ matrix and $\vec{v}$ and $\vec{w}$ vectors in $\mathbb{R}^m$. Prove that $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.
See Thm. 1.3.10 on p. 31.

2. (matrix equality) Let A be an $n \times p$ matrix and C and D be $p \times m$ matrices. Prove that $A(C + D) = AC + AD$.
Let $\vec{c_1}, \dots, \vec{c_m}$ be the columns of C and $\vec{d_1}, \dots, \vec{d_m}$ be the columns of D. Then (ith column of $A(C+D)$) $= A$(ith column of $C+D$) $= A(\vec{c_i} + \vec{d_i}) = A\vec{c_i} + A\vec{d_i} =$ (ith column of AC)+(ith column of AD) $=$ (ith column of $AC + AD$).

3. (matrix equality) Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Prove that the matrix of T is
$$A = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_m}) \end{bmatrix}.$$
See Thm. 2.1.2 on p. 47.

4. (linear transformation) Let $T(\vec{x})$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let c be a scalar in $\mathbb{R}$. Define $S(\vec{x}) = cT(\vec{x})$. Prove that $S(\vec{x})$ is a linear transformation.
We need to show two things: $S(\vec{v} + \vec{w}) = S(\vec{v}) + S(\vec{w})$ and $S(k\vec{v}) = kS(\vec{v})$.
Let $\vec{v}$, $\vec{w}$ in $\mathbb{R}^m$ and k be a scalar. Then:
$$S(\vec{v} + \vec{w}) = cT(\vec{v} + \vec{w}) = c(T(\vec{v}) + T(\vec{w})) = cT(\vec{v}) + cT(\vec{w}) = S(\vec{v}) + S(\vec{w})$$
and
$$S(k\vec{v}) = cT(k\vec{v}) = ckT(\vec{v}) = kcT(\vec{v}) = kS(\vec{v}).$$

5. (invertible) Let A and B be $n \times n$ matrices such that $BA = I_n$. Prove that A is invertible. (This is probably trickier than what you'll see on the exam.)
See Thm. 2.4.8. We need to show that $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$. Multiply both sides by B: $BA = I_n$ and $B\vec{0} = \vec{0}$, so $\vec{x} = \vec{0}$.

6. (invertible) Let A and B be invertible $n \times n$ matrices. Prove that AB is invertible.
   See Thm. 2.4.7. Or, verify that $(B^{-1}A^{-1})AB = I_n$, so that $(AB)^{-1} = B^{-1}A^{-1}$.

7. (subspace) Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then $\ker(T)$ is a subspace of $\mathbb{R}^n$.
   See Thm. 3.1.6. on p. 108.

8. (subspace) Let V and W be subspaces of $\mathbb{R}^n$. Define $V + W$ to be the set $\{\vec{v} + \vec{w} \mid \vec{v}$ is in V and $\vec{w}$ is in $W\}$. Determine whether $V + W$ is a subspace of $\mathbb{R}^n$.
   It is. We need to check that it contains $\vec{0}$, is closed under addition, and is closed under scalar multiplication.

   (i) As $\vec{0}$ is in both V and W, $\vec{0} = \vec{0} + \vec{0}$ is in $V + W$.

   (ii) Let $\vec{v_1} + \vec{w_1}$ and $\vec{v_2} + \vec{w_2}$ be two arbitrary elements of $V + W$. Then $(\vec{v_1} + \vec{w_1}) + (\vec{v_2} + \vec{w_2}) = (\vec{v_1} + \vec{v_2}) + (\vec{w_1} + \vec{w_2})$. $(\vec{v_1} + \vec{v_2})$ is in V and $(\vec{w_1} + \vec{w_2})$ is in W since V and W are closed under addition, so $(\vec{v_1} + \vec{w_1}) + (\vec{v_2} + \vec{w_2})$ is in $V + W$.

   (iii) Let $\vec{v} + \vec{w}$ be an arbitrary element of $V + W$ and k be a scalar. Then $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$. $k\vec{v}$ is in V and $k\vec{w}$ is in W since V and W are closed under scalar multiplication, so $k(\vec{v} + \vec{w})$ is in $V + W$.

3 Computational

1. Use Gauss-Jordan elimination to find all solutions of the system

$$\begin{array}{ccccccccc} 3x & + & 2y & - & 2z & - & w & = & 3 \\ x & + & y & + & z & + & 2w & = & 5 \\ & & 3y & - & 3z & - & 3w & = & 0. \end{array}$$

   Taking rref, we see: $x = 1 - 1/3t, y = 2 - 1/3t, z = 2 - 4/3t, w = t$.

2. Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -6 & 24 \\ 1 & -2 & 6 \\ -1 & 2 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 & 7 & 5 \\ -2 & 1 & 0 & -3 \\ 1 & 1 & 3 & 3 \end{bmatrix}.$$

(a) Find rref(A).

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\text{rref}(B) = I_3$,

$\text{rref}(C) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\text{rref}(D) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) Find rank(A).

Counting leading ones, $\text{rank}(A) = 2$, $\text{rank}(B) = 3$, $\text{rank}(C) = 1$, and $\text{rank}(D) = 2$.

(c) Is A invertible? If so, find A^{-1}. If not, explain how you know that it isn't.

A and C aren't invertible since they don't have rank 3. D isn't invertible since it isn't square.

$\text{rref}(\begin{bmatrix} B & | & I_n \end{bmatrix}) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \text{-}1 & 18/5 & 3/5 \\ 0 & 1 & 0 & \text{-}1/2 & 8/5 & 3/5 \\ 0 & 0 & 1 & 0 & 1/10 & 1/10 \end{array}\right]$, so $B^{-1} = \begin{bmatrix} -1 & 18/5 & 3/5 \\ -1/2 & 8/5 & 3/5 \\ 0 & 1/10 & 1/10 \end{bmatrix}$.

(d) Find a basis of im(A) and ker(A) and compute their dimensions.

Use rref(A). (Add a column of zeroes to get the augmented matrix.)

For the image, we just take the columns that have a leading one. Thus, a basis of im(A) is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. (That is, $\text{im}(A) = \text{span}\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\}$ and these vectors are linearly independent.) Thus, $\dim(\text{im}(A)) = 2$. Similarly,

im(B) has basis $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 24 \\ 6 \\ -4 \end{bmatrix}$,

im(C) has basis $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, and

im(D) has basis $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

Thus, $\dim(\text{im}(B)) = 3, \dim(\text{im}(C)) = 1, \dim(\text{im}(D)) = 2$.

(Compare these with the ranks.)

For the kernels, we find the solutions to $A\vec{x} = \vec{0}$ and use the free variables to write the general solution as the span of vectors. For A, we have general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix},$$

so a basis for $\ker(A)$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$. Thus, $\dim(\ker(A)) = 1$.

B is invertible, so $\ker(B) = \{\vec{0}\}$, so $\ker(B)$ has a basis with no vectors in it and thus $\dim(\ker(B)) = 0$.

For C, we have general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for $\ker(C)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\dim(\ker(C)) = 2$.

Similarly, a basis for $\ker(D)$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\dim(\ker(D)) = 2$.

(Note that in each case $\dim(\text{im}(A)) + \dim(\ker(A)) = \#$ of columns of A, as required by Rank-Nullity.)

(e) Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ (and $\vec{v_4}$ for D) be the columns of A. Are they linearly independent? If not, find a linear relation among them and use it to express one vector as a linear combination of the other two.

They are linearly independent if and only if $\text{rank}(A) = \#$ of columns of A. Thus, the columns of B are linearly independent and the columns of the others aren't. We can use a vector from the kernel to find a linear relation, and then rearrange it algebraically to find the required linear combinations.

For A, $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ is in $\ker(A)$, so $-\vec{v_1} - 2\vec{v_2} + \vec{v_3} = \vec{0}$ and so $\vec{v_3} = \vec{v_1} + 2\vec{v_2}$.

For C, $-2\vec{v_1} + \vec{v_2} + 0\vec{v_3} = \vec{0}$, and so $\vec{v_2} = 2\vec{v_1}$.

For D, $-\vec{v_1} - 2\vec{v_2} + \vec{v_3} + 0\vec{v_4} = \vec{0}$ and so $\vec{v_3} = \vec{v_1} + 2\vec{v_2}$.

(f) Are $\vec{v_1}, \vec{v_2}, \vec{v_3}$ (and $\vec{v_4}$ for D) a basis for $\mathbb{R}^3$?

$\dim(\mathbb{R}^3) = 3$, so any three linearly independent vectors form a basis. Thus, the column vectors of B form a basis of $\mathbb{R}^3$, while the column vectors of A, C, and D do not.

(g) Compute AB, BA, AC, etc. Compute $A\begin{bmatrix}1\\2\\3\end{bmatrix}$. Compute $A(2\vec{e_1}+3\vec{e_3})$.

Use `http://kinetigram.com/mck/LinearAlgebra/JPaisMatrixMult04/classes/JPaisMatrixMult04.html` to check your work.

3. Let $A=\begin{bmatrix}1 & 4\\2 & -1\end{bmatrix}$. Find all matrices which commute with A.

Write $B=\begin{bmatrix}a & b\\c & d\end{bmatrix}$. Then the equality $AB=BA$ can be expressed as

$$\begin{bmatrix}a+4c & b+4d\\2a-c & 2b-d\end{bmatrix}=\begin{bmatrix}a+2b & 4a-b\\c+2d & 4c-d\end{bmatrix}.$$

This gives us the four equations $a+4c=a+2b$, $b+4d=4a-b$, $2a-c=c+2d$, and $2b-d=4c-d$. Either solve for a, b, c, and d by Gauss-Jordan, or proceed by inspection (e.g., the first and last equations show $b=2c$) to find that all matrices of the form $\begin{bmatrix}a & b\\b/2 & a-b/2\end{bmatrix}$ commute with A.

(Alternatively, if you solve for b and d instead of c and d, you'll get the slightly nicer looking–but equivalent–form $\begin{bmatrix}a & 2c\\c & a-c\end{bmatrix}$.)

4. Find the matrices of the linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ which represent:

(a) a scaling by a factor of 3.

$\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$

(b) a 30° counterclockwise rotation.

$\begin{bmatrix}\sqrt{3}/2 & -1/2\\1/2 & \sqrt{3}/2\end{bmatrix}$

(c) a 45° clockwise rotation. (*Hint:* use a negative angle)

$\begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\-\sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}$

(d) a reflection about the line spanned by $\begin{bmatrix}-3\\4\end{bmatrix}$.

The unit vector $\vec{u}=\begin{bmatrix}-3/5\\4/5\end{bmatrix}$ spans this line (call it L). Then the transformation $\text{proj}_L(\vec{x})$ has matrix

$$\frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix},$$

so $\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}$ has matrix

$$\frac{1}{25}\begin{bmatrix} -7 & -24 \\ -24 & 7 \end{bmatrix}.$$

$(-7/25 = 2(9/25) - 1,\ -24/25 = 2(-12/25)$, etc.)

(e) an orthogonal projection onto the line spanned by $\begin{bmatrix} 5 \\ 12 \end{bmatrix}$.

(*Hint:* $5^2 + 12^2 = 13^2$)

The unit vector $\vec{u} = \begin{bmatrix} 5/13 \\ 12/13 \end{bmatrix}$ spans this line (call it L). Then the transformation $\text{proj}_L(\vec{x})$ has matrix

$$\frac{1}{169}\begin{bmatrix} 25 & 60 \\ 60 & 144 \end{bmatrix}.$$

(f) a vertical shear of strength 2 (that is, coming from a line with slope 2)

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

5. Interpret the linear transformations with the following matrices geometrically:

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

90° counterclockwise rotation

(b) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

horizontal shear of strength 3 (that is, coming from a line with slope 1/3)

(c) $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

30° counterclockwise rotation and a scaling by a factor of 2