The exam will cover Section 1.1-Section 3.3 and be composed approximately as follows:

True/False	30%
Definitions	10%
Proofs	10%
Computational	50%

In each section below, I'll explain what I mean by the category labels and then give a few examples of what a problem in that category might look like.

1 True/False

You will get a list of approximately six true/false questions with varying levels of difficulty, primarily on the more theoretical concepts. Some may come directly from statements of theorems, while others may require you to extrapolate slightly from a theorem or to think conceptually. You do not need to show any work on these. By "true," I will mean "always true." Thus, if something is sometimes true and sometimes false, you should answer false. For example, the statement "If A and B are $n \times n$ matrices, then $AB = BA$" is false since this is not always the case, even though this is true for certain matrices. If a statement begins "there exists," then it is just asking whether the property is true for at least one case, so for example the statement "There exists an $n \times n$ matrix A such that $A^2 = A$" is true, since I_n is one such matrix. Thinking geometrically may help you to find examples for "there exist" questions.

To prepare for these, make sure you know all of the theorems we've covered (the statements, not the proofs) and understand the geometric interpretations from Section 2.2.

Examples
True or false:

1. If a system of equations has fewer equations than unknowns, then it has infinitely many solutions.
2. If A in an $n \times m$ matrix, then $\text{rank}(A) \leq n$.
3. If A in an $n \times n$ matrix and $A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.
4. If a square matrix has two equal columns, then it is not invertible.

5. If a square matrix has two equal rows, then it is not invertible.
6. There exists a 2×2 matrix A such that $\text{rank}(A) = 0$.
7. There exists a 2×2 matrix A such that $\text{rank}(A) = 4$.
8. If A and B are $n \times n$ matrices, then $(AB)^2 = A^2B^2$.
9. For all $n \times n$ matrices A, B, and C, $(AB)C = A(BC)$.
10. A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a^2 + b^2 = 1$ must be invertible.
11. There exists a 2×2 matrix A such that $A^3 = I_2$ but $A \neq I_2$.
12. There exists a 2×2 matrix A such that $A^4 = I_2$ but $A^2 \neq I_2$.
13. There exists a 2×2 matrix A such that $A^2 = I_2$ but $A^4 \neq I_2$.
14. If A is a 3×4 matrix, then $A\vec{x} = \vec{0}$ has infinitely many solutions.
15. The solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{e_1}$ form a line in $\mathbb{R}^2$.
16. Let A and B be $n \times n$ matrices. If $\vec{v}$ is in $\ker(B)$, then $\vec{v}$ is in $\ker(AB)$.
17. Let A and B be $n \times n$ matrices. If $\vec{v}$ is in $\ker(B)$, then $\vec{v}$ is in $\ker(BA)$.
18. If $\vec{x} = \vec{v}$ and $\vec{x} = \vec{w}$ are two solutions to $A\vec{x} = \vec{b}$, then $\vec{x} = \vec{v} + \vec{w}$ is a solution too.
19. If $\vec{v}$ and $\vec{w}$ are in $\text{im}(A)$, then $2\vec{v} - 7\vec{w}$ is in $\text{im}(A)$ too.
20. If $A\vec{v} = A\vec{w}$, then $\vec{v} - \vec{w}$ is in $\ker(A)$.
21. Let A be an $n \times m$ matrix. Then $\text{im}(A)$ is a subspace of $\mathbb{R}^n$.
22. Let A be an $n \times m$ matrix. Then $\text{im}(A)$ is a subspace of $\mathbb{R}^m$.
23. If $\ker(A) = \{\vec{0}\}$ for an $n \times m$ matrix A, then $n \leq m$.
24. If A is an upper-triangular matrix, then A is invertible.
25. If A is a permutation matrix, then $A\vec{e_1} = \vec{e_i}$ for some i.
26. Let A be a $n \times m$ matrix. Then $\dim(\text{im}(A)) + \dim(\ker(A)) = n$.
27. Sec. 2.4: 67-75
28. The chapter reviews

2 Definitions

You will either be asked to define a term or two. Your definitions must be precise and accurate to receive full credit. It's not necessary to use the exact words that the book does, as long as your definition is equivalent. You should know anything that the book labels as a "Definition" as well as *reduced row-echelon form* (p. 16).

Examples
Question: Define the **kernel** of an $n \times m$ matrix A.
Answer: The kernel of A is the set of all solutions of the equation $A\vec{x} = \vec{0}$.

Question: Define what it means to say that $\vec{b}$ is a **linear combination** of the vectors $\vec{v_1}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$.
Answer: We say that a vector $\vec{b}$ in $\mathbb{R}^n$ is a linear combination of the vectors $\vec{v_1}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$ if there exist scalars $x_1, \ldots, x_m$ in $\mathbb{R}$ such that

$$\vec{b} = x_1\vec{v_1} + \cdots + x_m\vec{v_m}.$$

Note that the following terms are defined in the text but not labeled as definitions: square matrix, diagonal matrix, upper/lower triangular matrix (p. 9), homogeneous system (p.35, #47), standard vectors (p. 47), all of the geometric terminology from Section 2.2 (projections, reflections, etc.), permutation matrix (p. 89, #42) hyperplane (p. 134, #33). I won't ask you to define these, but you should know what these words mean since they may be used in other questions on the exam.

3 Proofs

You will be asked to provide a proof of something. This will be something which can be proven in a fairly straightforward manner and without being too long. It is possible that you will be asked to prove a theorem seen in class, but memorizing proofs from the book is probably not a good use of your time. Rather, make sure you understand and know how to use our techniques for showing:

1. that two vectors/matrices are equal (by showing the components/columns are equal).
2. that a transformation is linear (by showing $T(\vec{v}+\vec{w}) = T(\vec{v})+T(\vec{w})$ and $T(k\vec{v}) = kT(\vec{v})$).
3. that a matrix is invertible (see Summary 3.3.10; one technique is to show that $A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$; another is to find the inverse explicitly).

4. that a given set is or is not a subspace of $\mathbb{R}^n$ (by applying Definition 3.2.1).

You may use any theorems we've seen in your proof (unless you're being asked to prove the theorem itself, of course).

Examples

1. (vector equality) Let A be an $n \times m$ matrix and $\vec{v}$ and $\vec{w}$ vectors in $\mathbb{R}^m$. Prove that $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.

2. (matrix equality) Let A be an $n \times p$ matrix and C and D be $p \times m$ matrices. Prove that $A(C + D) = AC + AD$.

3. (matrix equality) Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Prove that the matrix of T is
$$A = \begin{bmatrix} T(\vec{e_1}) & \dots & T(\vec{e_m}) \end{bmatrix}.$$

4. (linear transformation) Let $T(\vec{x})$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let c be a scalar in $\mathbb{R}$. Define $S(\vec{x}) = cT(\vec{x})$. Prove that $S(\vec{x})$ is a linear transformation.

5. (invertible) Let A and B be $n \times n$ matrices such that $BA = I_n$. Prove that A is invertible. (This is probably trickier than what you'll see on the exam.)

6. (invertible) Let A and B be invertible $n \times n$ matrices. Prove that AB is invertible.

7. (subspace) Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then $\ker(T)$ is a subspace of $\mathbb{R}^n$.

8. (subspace) Let V and W be subspaces of $\mathbb{R}^n$. Define $V + W$ to be the set $\{\vec{v} + \vec{w} \mid \vec{v} \text{ is in } V \text{ and } \vec{w} \text{ is in } W\}$. Determine whether $V + W$ is a subspace of $\mathbb{R}^n$.

(I probably won't ask you to prove one of those specific examples, so be sure you're focusing on how the techniques are used and not on the details.)

4 Computational

These will focus on applying the various techniques we've seen to find numerical answers to problems. In particular, you may be asked to:

1. Solve a system of linear equations (Sec. 1.1, #1-10, Sec. 1.2, #1-12)

2. Row-reduce a matrix (Gauss-Jordan elimination) (Sec. 1.2, #1-12); compute its rank (Sec. 1.3, #2-4)

3. Put a system of equations in matrix form (Sec. 1.3, #9)
4. Matrix/vector algebra (sums, products, dot products, scalar multiples, *etc.*) (Sec. 1.3, #10-20, Sec. 2.3, #1-14)
5. Find the matrices which commute with a given matrix. (Sec. 2.3, #17-26)
6. Find the inverse of a matrix/linear transformation (Sec. 2.4, #1-20)
7. Find the matrices of linear transformations defined geometrically; interpret a given matrix/linear transformation geometrically (Sec. 2.2)
8. Find a basis for the image and kernel of a linear transformation; determine their dimensions (Sec. 3.3, #1-25)
9. Determine if a set of vectors is linearly independent; find linear relations between vectors; express one vector as a linear combination of others (Sec. 3.2, #10-26)
10. Determine if a set of vectors is a basis for $\mathbb{R}^n$ (Sec. 3.3, #27-28)

Note that you can check your answers to many of these. After solving a system of linear equations, plug your answer back in to see if it works. (If you're taking $\text{rref}(A)$ for some reason other than solving a system and so don't have an augmented matrix, you can still check your work this way since $\text{rref}(\left[A \mid \vec{0}\right]) = \left[\text{rref}(A) \mid \vec{0}\right]$.) After you find the matrices that commute, take both products (AB and BA) to verify that they're the same. After finding the inverse, verify that $AA^{-1} = I_n$. After finding a matrix with a geometric interpretation, check that it does what it's supposed to on a few vectors (such as $\vec{e_1}$ and $\vec{e_2}$). After finding the kernel, make sure the vectors in it really are solutions to $A\vec{x} = \vec{0}$. Use vectors in the kernel to look for linear relations in the image.

Examples

You should be able to create your own examples easily by making up your own matrices/systems/etc. (or use the exercises in the book above), but a few follow nonetheless.

1. Use Gauss-Jordan elimination to find all solutions of the system

$$\begin{array}{ccccccccc} 3x & + & 2y & - & 2z & - & w & = & 3 \\ x & + & y & + & z & + & 2w & = & 5 \\ & & 3y & - & 3z & - & 3w & = & 0. \end{array}$$

2. Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -6 & 24 \\ 1 & -2 & 6 \\ -1 & 2 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}, D = \begin{bmatrix} 1 & 3 & 7 & 5 \\ -2 & 1 & 0 & -3 \\ 1 & 1 & 3 & 3 \end{bmatrix}.$$

(a) Find $\operatorname{rref}(A)$.

(b) Find $\operatorname{rank}(A)$.

(c) Is A invertible? If so, find A^{-1}. If not, explain how you know that it isn't.

(d) Find a basis of $\operatorname{im}(A)$ and $\ker(A)$ and compute their dimensions.

(e) Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ (and $\vec{v_4}$ for D) be the columns of A. Are they linearly independent? If not, find a linear relation among them and use it to express one vector as a linear combination of the other two.

(f) Are $\vec{v_1}, \vec{v_2}, \vec{v_3}$ (and $\vec{v_4}$ for D) a basis for $\mathbb{R}^3$?

(g) Same questions for B, C, and D.

(h) Compute AB, BA, AC, etc. Compute $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Compute $A(2\vec{e_1} + 3\vec{e_3})$.

3. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$. Find all matrices which commute with A.

4. Find the matrices of the linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ which represent:

(a) a scaling by a factor of 3.

(b) a 30° counterclockwise rotation.

(c) a 45° clockwise rotation. (*Hint:* use a negative angle)

(d) a reflection about the line spanned by $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

(e) an orthogonal projection onto the line spanned by $\begin{bmatrix} 5 \\ 12 \end{bmatrix}$.

(*Hint:* $5^2 + 12^2 = 13^2$)

(f) a vertical shear of strength 2 (that is, coming from a line with slope 2)

5. Interpret the linear transformations with the following matrices geometrically:

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$