Section 7.1

1. Suppose $A\vec{v} = \lambda\vec{v}$. Then $A^3\vec{v} = A^2(A\vec{v}) = A^2(\lambda\vec{v}) = \lambda A^2\vec{v} = \cdots = \lambda^3\vec{v}$. Thus, $\vec{v}$ is an eigenvector of A^3 with eigenvalue λ^3. Work for the rest of #1-6 is similar.

2. Yes; eigenvalue $\frac{1}{\lambda}$.

4. Yes; eigenvalue 7λ.

6. Suppose $A\vec{v} = \lambda\vec{v}$ and $B\vec{v} = \mu\vec{v}$. Then $AB\vec{v} = A\mu\vec{v} = \mu A\vec{v} = \mu\lambda\vec{v}$, so $\vec{v}$ is an eigenvector of AB (with eigenvalue $\mu\lambda$.)

11. Solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for a, b, c, d.

15. Let A be the matrix of this transformation. If $\vec{v}$ is any vector on L, it will reflect to itself, so $A\vec{v} = \vec{v}$ and so $\vec{v}$ is an eigenvector with eigenvalue 1. If $\vec{w}$ is perpendicular to L (that is, $\vec{w}$ is in $L^\perp$), it will reflect straight across L, so $A\vec{w} = -\vec{w}$ and so $\vec{w}$ is an eigenvector with eigenvalue -1. It's clear geometrically that no other vectors will map to a scalar multiple of themselves, so these are the only eigenvectors and eigenvalues. To find an eigenbasis, pick one vector from L and one vector from $L^\perp$.

43. Suppose that the eigenvalue corresponding to $\vec{e_i}$ is λ_i for $i = 1\ldots, n$. Note $A = AI_n = A\begin{bmatrix}\vec{e_1}\ldots\vec{e_n}\end{bmatrix} = \begin{bmatrix}A\vec{e_1}\ldots A\vec{e_n}\end{bmatrix} = \begin{bmatrix}\lambda_1\vec{e_1}\ldots\lambda_n\vec{e_n}\end{bmatrix}$. By varying the eigenvalues, we can get out any diagonal matrix, so V is the space of diagonal matrices and has dimension n (since there are n diagonal entries, corresponding to the n choices $\lambda_1, \ldots, \lambda_n$.

50. As in our first example in class, parts (a) and (b) will let us find an eigenbasis, which we use in part (c).

    (a) Set $\vec{v_0} = \vec{v}(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Check that $A\vec{v_0} = 2\vec{v_0}$ so, $\vec{v_0}$ is an eigenvector with eigenvalue 2. Then $\vec{v}(t) = A^t\vec{v_0} = 2^t\vec{v_0}$. Thus, $h(t) = 100(2^t)$ and $f(t) = 100(2^t)$. (Thus, the population of hares and foxes will increase exponentially.)

    (b) Set $\vec{v_0} = \vec{v}(0) = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$. Proceeding as above, we see that $\vec{v_0}$ is an eigenvector with eigenvalue 3 and $\vec{v}(t) = 3^t\vec{v_0}$, so $h(t) = 200(3^t)$ and $f(t) = 100(3^t)$. (Again, both populations increase exponentially.)

(c) Set $\vec{v_0} = \vec{v}(0) = \begin{bmatrix} 600 \\ 500 \end{bmatrix}$. The eigenvectors in the previous two parts form an eigenbasis and we find (using Section 3.4 techniques or inspection) that $\begin{bmatrix} 500 \\ 700 \end{bmatrix} = 4 \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 200 \\ 100 \end{bmatrix}$. Thus, we find $\vec{v}(t) = 4(2^t) \begin{bmatrix} 100 \\ 100 \end{bmatrix} + (3^t) \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ and so $h(t) = 400(2^t) + 200(3^t)$ and $f(t) = 400(2^t) + 100(3^t)$. (Again, both populations increase exponentially.)

Section 7.2

22. By Theorem 6.2.1, $\det(A - \lambda I_n) = \det((A - \lambda I_n)^{\mathrm{T}}) = \det(A^{\mathrm{T}} - \lambda I_n)$, so both A and A^{T} have the same characteristic polynomial, and so the same eigenvalues with the same algebraic multiplicities.

38. By Theorem 7.2.4, the characteristic polynomial of A is $f_A(\lambda) = \lambda^2 - 5\lambda - 14$. Factoring, we have eigenvalues $\lambda = -2$ and $\lambda = 7$. (Note that this technique only worked since we were dealing with a 2×2 matrix: if the matrix were larger, we wouldn't have been able to find all of the coefficients of the characteristic polynomial in this way.)

40. By direct computation,

$$\begin{aligned} \mathrm{tr}(AB) &= (\text{sum of all products of the form } a_{ij}b_{ji}) \\ &= (\text{sum of all products of the form } b_{ji}a_{ij}) = \mathrm{tr}(BA). \end{aligned}$$

41. Write $B = S^{-1}AS$ for some matrix S and use #40: $\mathrm{tr}(B) = \mathrm{tr}(S^{-1}AS) = \mathrm{tr}((S^{-1}A)S) = \mathrm{tr}(S(S^{-1}A)) = \mathrm{tr}(A)$.

43. Take the trace of both sides. By #40, the trace of the left-hand side will be 0, but the trace of the right-hand side is n. This is a contradiction, so no such matrices exist.

44. *Hint:* There are at least a couple of ways to do this; try to use either #41 or #43. The matrices A and B in #41 or #43 may not be the same as the matrices A and B in #44.

Section 7.3

28. Note $J_n(k)$ is triangular, so its diagonal entries are its eigenvalues. Thus, the only eigenvalue is k. Also $E_k = \ker(J_n(k) - kI_n) = \mathrm{span}\{\vec{e_1}\}$ (all of the other columns are linearly independent since each has a leading 1), so the geometric multiplicity is 1 and the algebraic multiplicity is n.

32. The geometric multiplicity of λ as an eigenvalue of A is

$$\dim(\ker(A - \lambda I_n)) = n - \operatorname{rank}(A - \lambda I_n)$$

by Rank-Nullity.

The geometric multiplicity of λ as an eigenvalue of A^{T} is

$$\begin{aligned}\dim(\ker(A^{\mathrm{T}} - \lambda I_n)) &= \dim(\ker((A - \lambda I_n)^{\mathrm{T}})) \\ &= n - \operatorname{rank}((A - \lambda I_n)^{\mathrm{T}}) \\ &= n - \operatorname{rank}(A - \lambda I_n)\end{aligned}$$

by Rank-Nullity and Theorem 5.3.9c. Thus, the two geometric multiplicities are equal.

Section 7.4

37. Yes, $\lambda^2 - 7\lambda + 7$ is the characteristic polynomial for both, so they have the same real eigenvalues $\lambda_{1,2} = \dfrac{7 \pm \sqrt{21}}{2}$ and so are both similar to the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (by Theorem 7.4.4) and so A is similar to B by Theorem 3.4.6 (parts b and c).

38. No! For example, consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

But wait! Isn't this the same problem as #37 with the numbers changed?
Not quite. Note that $\lambda^2 - 7\lambda + 7$ had two distinct roots (and so two distinct eigenvalues) while $\lambda^2 - 4\lambda + 4$ has only one distinct root (and so the eigenvalue 2 has algebraic multiplicity 2, but in the counterexample given above it has a geometric multiplicity of only 1).

Section 7.5

12. First verify that $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \bar{z}\bar{w}$, and $\overline{(z^n)} = \overline{z}^n$. (These are easy to check, from the definitions.)

Suppose that λ_0 is a complex root of $f(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$ where the coefficients $a_n, \ldots, a_0$ are real. Then $a_n\lambda_0^n + \cdots + a_1\lambda_0 + a_0 = 0$ and the result follows by taking the conjugate of both sides and using the above properties. (Note that since a_i is real, $\overline{a_i} = a_i$.)

27. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A (repeated according to algebraic multiplicity, so $\lambda_1 = \lambda_2 \neq \lambda_3$). Then $\operatorname{tr}(A) = 1 = 2\lambda_2 + \lambda_3$ and $\det(A) = 3 = \lambda_2^2\lambda_3$. Solving for λ_2, λ_3, we see $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 3$.

42. (Note that this problem shows that complex eigenvalues and eigenvectors of real matrices always come in conjugate pairs. This is often useful. Compare this to #12 applied to the characteristic polynomial.)

(a) Recall $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \bar{z}\bar{w}$ from #12. Then the ijth entry of $\overline{AB}$ is

$$\overline{\sum_{k=1}^{p} a_{ik}b_{kj}} = \sum_{k=1}^{p} \overline{a_{ik}b_{kj}} = \sum_{k=1}^{p} \overline{a_{ik}}\,\overline{b_{kj}},$$

which is the ijth entry of $\bar{A}\bar{B}$.

(b) We will use part (a), where B is the $n \times 1$ matrix $\vec{v} + \imath\vec{w}$. Let $\lambda = p + \imath q$. Then $AB = \lambda B$ and so $A\overline{B} = \overline{AB} = \overline{\lambda B} = \bar{\lambda}\overline{B}$, so $A(\vec{v} - \imath\vec{w}) = (p - \imath q)(\vec{v} - \imath\vec{w})$.