Section 6.1

43. Note that every entry of $-A$ is a corresponding entry of A multiplied by -1. Since each pattern P contains n entries of A, a pattern P' obtained from the same entries as P will have $\text{prod}(P') = (-1)^n \text{prod}(P)$ and so $\det(-A) = (-1)^n \det(A)$. (Note you can also do this easily with the Section 6.2 material, since we know what the effect of multiplying a row by a scalar is.)

57. There is only one nonzero pattern P in a permutation matrix. $\text{prod}(P) = 1$ and $\text{sgn}(P) = \pm 1$ (depending on the number of inversions), so for a permutation matrix A, $\det(A) = \pm 1$.

Section 6.2

31. (Feel free to just read this problem and not actually do it, if you want.)

    (a) If $n = 1$, then $A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix}$ and $\det(A) = a_1 - a_0$, so the product formula holds.

    (b) The idea here is a technique known as "proof by induction." We'll show that if it holds for $n-1$, then it holds for n also. Then, since we know it holds for $n = 1$ from the first part, this will show it holds for $n = 2$. Then, reapplying the result will show that it holds for $n = 3$, and so on, so that it in fact holds for all positive integers n.

    Suppose that the formula holds for $(n-1) \times (n-1)$ matrices and let $f(t)$ be as defined in the problem. Each pattern in the matrix has one and only one component that involves the variable t, each time to a power at most n. Since some terms do contain t^n, f will be a polynomial of degree n unless all of the t^n terms cancel out. But, the coefficient of t^n will be determined by patterns that have no components in the last row or column except for t^n: that is, patterns whose last component is t^n and other components come from the $n \times n$ submatrix with the last row and column removed. But this is the $(n-1)$st Vandermonde matrix, so by our assumption, the coefficient of t^n will be

    $$k = \prod_{n-1 \geq i > j} (a_i - a_j),$$

    which is nonzero since we chose distinct scalars $a_1, \ldots a_n$.

    If we let $t = a_0, a_1 \ldots, a_{n-1}$, then the matrix will have two equal columns and thus determinant 0. Thus, $f(a_0) = \cdots = f(a_{n-1}) = 0$ and since a degree

n polynomial has at most n roots, it follows from the Fundamental Theorem of Algebra (i.e., the polynomial factoring theorem) that

$$f(t) = k(t - a_0)(t - a_1)\dots(t - a_{n-1})$$

where k is the scalar we computed above. Letting $t = a_n$, the equation $\det(A) = f(a_n)$ is precisely Vandermonde's formula (since the terms missing in k are precisely those added by setting $t = a_n$ in $(t - a_0)(t - a_1)\dots(t - a_{n-1})$.)

32. Use #31 with $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$. Then the determinant is 288.

37. Apply Theorem 6.2.6 to the equation $AA^{-1} = I_n$, so $\det(A)\det(A^{-1} = 1$. Note that since A and A^{-1} have integers entries, their determinants will be integers (think about the patterns: each is a product of integers and so an integer; then, the determinant is a sum of integers and so an integer). The only way two integers can multiply to 1 is if they are either both 1 or both -1, so $\det(A) = \pm 1$.