Section 5.1

10. $\vec{u} \cdot \vec{v} = 2 + 3k + 4$. Thus $\vec{u} \cdot \vec{v} = 0$ if and only if $k = -2$.

15. *Hint:* We've previously seen that this will be a hyperplane, so we just need to find three linearly independent vectors perpendicular to $\vec{v}$ (since a hyperplane in $\mathbb{R}^4$ has dimension 3). Since we want $\vec{v} \cdot \vec{x} = x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ (with $\vec{x}$ defined in the obvious way), this is equivalent to finding a basis of the space $\ker(\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix})$.

Section 5.2

Note: Problems 1-14 and 15-28 are really the same problems, but one is asking about Gram-Schmidt and one is asking about QR. Since these are both really the same, you might want to do both for practice, in which case you can check your answers by adding 14 (so, for example, the answer to #1 is given in the back of the book as #1 and #15).

32. First, we want to find a basis, period. Then we'll apply Gram-Schmidt to find an orthonormal basis. A basis for this plane is just a basis for $\ker(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix})$. Or, we can just find a basis by inspection, since any two linearly independent vectors in the plane will form a basis. Either way, we end up with $\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. (Or something else: other starting bases are possible, in which case you may end up with a different orthonormal basis at the end.) By Gram-Schmidt, we end up with the basis $\vec{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

    Other solutions are possible if you started with a different $\vec{v_1}$ and $\vec{v_2}$.

Section 5.3

1. Not orthogonal. The column vectors aren't perpendicular.

5. Not orthogonal. The column vectors will have length 3, not 1.

7. Orthogonal by Theorem 5.3.4a.

11. Orthogonal. Note $A^{\mathrm{T}} = A^{-1}$ by Theorem 5.3.7, which is orthogonal by Theorem 5.3.4b.

21. Symmetric, since $(A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}A$.

23. Not necessarily symmetric (counterexamples are easy to find). In fact,

$$(A - A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} - A = -(A - A^{\mathrm{T}}),$$

so $A - A^{\mathrm{T}}$ is skew-symmetric. (We've used the fact that $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$, which we haven't formally proven, but this is obvious from the definition of the transpose.)

25. Symmetric, since $(A^{\mathrm{T}}B^{\mathrm{T}}BA)^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}BA$. (Check this.)

55. Let E_{ij} be the matrix with a 1 in the ijth position and 0's elsewhere. Then a basis for this space consists of the matrices with diagonal entries, $E_{11}, \ldots, E_{nn}$, as well as all matrices of the form $E_{ij} + E_{ji}$ where $i < j$. (If you don't see this, try looking at the cases $n = 2, 3, 4$ to see what's going on.) Thus, the dimension is equal to the number of entries on the diagonal plus the number of entries strictly above the diagonal. As the matrices are $n \times n$, there are n entries on the diagonal. There are n^2 entries total and so $n^2 - n$ entries not on the diagonal. Half are above and half are below, so there are $\frac{n^2-n}{2}$ entries strictly above the diagonal. Thus, the dimension of this space is $n + \frac{n^2-n}{2} = \frac{n^2+n}{2}$.

50. (*Note:* #50 and #51 were not assigned. However, since a question was raised about uniqueness of the QR factorization, I'm going to provide an answer to #50 nonetheless (and #51 is in the back of the book). Together, they show that QR factorization is *almost* unique (but that we need the extra restriction that the diagonal entries are positive, which is necessary since otherwise we could modify Q by multiplying a column by -1 and changing R in a corresponding way.)

   (a) As A is orthogonal, $A^{\mathrm{T}} = A^{-1}$. As A is upper-triangular, A^{-1} is lower-triangular (since A^{T} is) and also upper-triangular (since the inverse of an upper-triangular matrix is upper-triangular; see Section 2.4 #35, also not assigned). To be both upper- and lower-triangular, A^{-1} must be diagonal and since $A^{\mathrm{T}} = A^{-1}$, it must be that A is diagonal also, so $A^{-1} = A$. But the columns of A are unit vectors (as A is orthogonal), so since the entries are all positive, it must be that $A = I_n$.

   (b) Note $Q_2^{-1}Q_1$ is orthogonal by Theorem 5.3.4 and $R_2R_1^{-1}$ is upper-triangular with positive diagonal entries. Thus, $Q_2^{-1}Q_1 = R_2R_1^{-1}$ is orthogonal and upper-triangular with positive diagonal entries and so must be equal to I_n. Thus, $Q_1 = Q_2$ and $R_1 = R_2$.

Section 5.4

4. By Theorem 5.4.1, $(\mathrm{im}(B))^{\perp} = \ker(B^{\mathrm{T}})$ for any matrix B. Do this for $B = A^{\mathrm{T}}$ and the result follows by taking $\perp$ of both sides and applying Theorem 5.1.8d.

7. Note $A^{\mathrm{T}} = A$, so $\ker(A^{\mathrm{T}}) = \ker(A)$. Applying Theorem 5.4.1, we see $(\mathrm{im}(A))^{\perp} = \ker(A)$. Thus, $\mathrm{im}(A)$ and $\ker(A)$ are orthogonal complements.

Section 5.5

2. Yes. Note $\langle f, g+h\rangle = \langle g+h, f\rangle = \langle g, f\rangle + \langle h, f\rangle = \langle f, g\rangle + \langle f, h\rangle$.

4. Both are the dot product.

7. Properties (a)-(c) hold for any k (verify this). Since $\langle v, v\rangle$ is positive for $v \neq 0$, property (d) holds if and only if $k > 0$. Thus, this is an inner product if and only if $k > 0$.

8. It is a linear transformation by properties (b) and (c). If $w = 0$, then $\mathrm{im}(T) = \{0\}$ and $\ker(T) = V$. If $w \neq 0$, then $\mathrm{im}(T) = \mathbb{R}$ and $\ker(T)$ consists of all vectors perpendicular to w. (Compare this to #35 in Section 3.1.)

17. Properties (a)-(c) hold for any T (verify this). Since $\langle v, v\rangle = \|T(v)\|^2$, (d) will hold if and only if $T(v) \neq 0$ for all $v \neq 0$, which is to say $\ker(T) = \{0\}$.

22. Let $g(t) = 1$ and apply the Cauchy-Schwarz inequality to $f(t)$ and $g(t)$. Since $\|g(t)\| = 1$, this tells us that $|\langle f, g\rangle| \leq \|f\|$, so squaring both sides gives us $\langle f, g\rangle^2 \leq \|f\|^2$, which means that

$$\left(\int_0^1 f(t)\mathrm{d}t\right)^2 \leq \int_0^1 (f(t))^2\mathrm{d}t.$$

Note: We got a fairly sophisticated calculus result using only tools from algebra. Pretty cool, no?

26. $a_0 = 0$.

$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\mathrm{d}t = \frac{1}{\pi}\left(-\int_{-\pi}^{0}\sin(kt)\mathrm{d}t + \int_0^{\pi}\sin(kt)\mathrm{d}t\right) = \frac{2}{\pi}\int_0^{\pi}\sin(kt)\mathrm{d}t$, which is 0 if k is even and $\frac{4}{\pi k}$ if k is odd.

$c_k = 0$ for all k since the integrand is an odd function.

So, for example $f_6(t) = \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t) + \frac{4}{5\pi}\sin(5t)$. For a graph, go to `www.wolframalpha.com` and type in "plot 4/pi sin(t)+4/(3 pi) sin(3t) + 4/(5 pi) sin(5t) on -pi < t < pi" (without the quotation marks). As we've mentioned, the Fourier approximations are only valid on $[-\pi, \pi]$: try "plot 4/pi sin(t)+4/(3 pi) sin(3t) + 4/(5 pi) sin(5t)" to see what this approximation looks like outside of this domain: as expected (since $\sin(kt)$ is periodic), it just repeats over and over again.