Section 4.1

1. Not a subspace. Doesn't contain the zero polynomial and not closed under addition or scalar multiplication.

2. Is a subspace. If we write $p(t) = a + bt + ct^2$, then the condition $p(2) = 0$ tells us that $a + 2b + 4c = 0$, or $a = -2b - 4c$, so that a general element is of the form $p(t) = (-2b - 4c) + bt + ct^2 = b(t-2) + c(t^2 - 4)$. Verify that $t - 2$, $t^2 - 4$ are linearly independent, so that $t - 2$, $t^2 - 4$ is a basis.

4. Is a subspace. If we write $p(t) = a + bt + ct^2$, then the condition $\int_0^1 p(t)\mathrm{d}t = 0$ tells us that $a + \frac{b}{2} + \frac{c}{3} = 0$, or $a = -\frac{b}{2} - \frac{c}{3}$. Proceeding as above, a basis is $t - \frac{1}{2}$, $t^2 - \frac{1}{3}$.

9. Not a subspace. It's not closed under multiplication by a negative scalar.

17. Let E_{ij} be the matrix with a 1 in the (i, j)th position and 0's elsewhere. Then the set of E_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq m$ form a basis of $\mathbb{R}^{n \times m}$, so that the dimension is nm.

25. Proceed as in #2.

45. A basis is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, $\dim(V) = 3$.

Section 4.2

Note: While the problems don't ask you to, I recommend also finding kernels and images on #3, 7, 20, 29, and 45 for practice.

3. Linear; not an isomorphism as $\dim(\mathbb{R}^{2\times 2}) \neq \dim(\mathbb{R})$.
   $\ker(T) = \text{span}\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$.
   $\text{im}(T) = \text{span}\{1\}$.

7. Linear and an isomorphism. Since it's an isomorphism, $\ker(T) = \{0\}$ and $\text{im}(T) = \mathbb{R}^{2\times 2}$.

20. Linear and an isomorphism (in fact, it is its own inverse). Thus, $\ker(T) = \{0\}$ and $\text{im}(T) = \mathbb{C}$.

29. Linear; not an isomorphism since all constant functions map to 0 (and hence the map is not invertible). Write $f(t) = a + bt + ct^2$, so $f'(t) = b + 2ct$. To be in the kernel, $f'(t) = 0$ for all t, so $b = c = 0$. Thus, $\ker(T) = \text{span}\{1\}$. From the form of $f'(t)$, we see that we can get all polynomials of degree ≤ 1 out, so $\text{im}(T) = \text{span}\{1, t\} = \mathcal{P}_1$.

45. Linear; not an isomorphism since the constant functions aren't in the image. $\ker(T) = \{0\}$, $\text{im}(T) = \{f(t) \text{ in } \mathcal{P} \mid f(0) = 0\}$. We haven't defined the span of infinitely many vectors, but if we had, then $\text{im}(T)$ would be $\text{span}\{t, t^2, t^3, t^4, \dots\}$.

Section 4.3

5. Invertible; so an isomorphism.

15. Invertible; so an isomorphism.

16. $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Invertible; so an isomorphism.

21. Invertible; so an isomorphism.

(See the problems from Section 4.2 for some examples of non-isomorphisms.)