Section 3.1

44. (a) Suppose $\text{rref}([A|\vec{b_1}]) = [B|\vec{b_2}]$. We've seen $B\vec{x} = \vec{b_2}$ has the same solutions as $A\vec{x} = \vec{b_1}$ (since row-operations respect the notion of equality), so in particular, setting $\vec{b_1} = \vec{0}$ (so that $\vec{b_2} = \vec{0}$ too) tells us that $\ker(A) = \ker(B)$.

    (b) These are not always equal. Note that the image depends on the order of the equations, which is not respected when taking rref. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

    Then $\text{im}(A) = \text{span}(\vec{e_2})$, but $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and so $\text{im}(B) = \text{span}(\vec{e_1})$.

Section 3.2

1. This one isn't a subspace for many reasons. For example, note that $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in W, but $2\vec{v}$ isn't and $\vec{v} + \vec{v}$ isn't. Also, $\vec{0}$ isn't in W, so in fact W fails all three properties (though it only needs to fail one to not be a subspace).

2. This one is "really close" to being a subspace, but it isn't closed under multiplication by a negative scalar. For example, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in W, but $-\vec{v}$ isn't.

3. This is a subspace by Thm. 3.2.2.

8. For example, $\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

42. Let $c_1\vec{v_1} + \cdots + c_m\vec{v_m} = \vec{0}$. We wish to show that all of the scalars $c_i = 0$. For any i, consider the dot product

$$(c_1\vec{v_1} + \cdots + c_m\vec{v_m}) \cdot \vec{v_i} = \vec{0} \cdot \vec{v_i},$$

which simplifies to

$$c_1(\vec{v_1} \cdot \vec{v_i}) + \cdots + c_m(\vec{v_m} \cdot \vec{v_i}) = 0.$$

Since the v_i are perpendicular unit vectors, $\vec{v_j} \cdot \vec{v_i} = 0$ whenever $i \neq j$ and $\vec{v_i} \cdot \vec{v_i} = 1$ for any i. Thus, the above simplifies to $c_i = 0$ and so repeating this for $i = 1, \ldots, m$, we see that $c_1 = \cdots = c_m = 0$ and so we have only the trivial relation among the vectors $\vec{v_1}, \ldots \vec{v_m}$ and so they are linearly independent.

Section 3.3

33. Let $A = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}$ (a $1 \times n$ matrix). Then $V = \ker(A)$. At least one of the c_i is nonzero, so $\text{rank}(A) = 1$. By the Rank-Nullity Theorem, $\dim(V) = n - 1$. A hyperplane in $\mathbb{R}^3$ is a plane and a hyperplane in $\mathbb{R}^2$ is a line. (This is a good problem to remember; hyperplanes are common and this result will save you time in calculating their dimension.)

35. Case in point, if $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and we let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then all vectors $\vec{x}$ are perpendicular to $\vec{v}$ if and only if they satisfy the equation $\vec{v} \cdot \vec{x} = 0$, which can also be expressed as $v_1 x_1 + \cdots + v_n x_n = 0$. This is a hyperplane and $\vec{v} \neq \vec{0}$, so by #33, the dimension of this space is $n - 1$.

78. Suppose that V is an m-dimensional space with basis $\vec{v_1}, \dots, \vec{v_m}$. As they are a basis, they are in particular linearly independent. Then, if $\vec{w_1}, \dots, \vec{w_q}$ is any set of vectors which spans V, it follows that $q \geq m$ by Thm. 3.3.1.

Section 3.4

37. We want to find a basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}\}$ such that $T(\vec{v_1}) = a\vec{v_1}$ and $T(\vec{v_2}) = b\vec{v_2}$ for some scalars a and b, as then the $\mathcal{B}$-matrix of T will be

$$B = \begin{bmatrix} [T(\vec{v_1})]_{\mathcal{B}} & [T(\vec{v_2})]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

which is diagonal.

One strategy is to note that $T(\vec{v}) = \vec{v} = 1\vec{v}$ for any $\vec{v}$ parallel to the line L onto which we project and $T(\vec{w}) = \vec{0} = 0\vec{w}$ for any $\vec{w}$ perpendicular to L. Thus, we can find a basis with the desired properties by picking one vector parallel to L and one vector perpendicular to L. For example, $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \}$.

In this case, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.