Section 2.2, part 2

13. We know that $\text{proj}_L(\vec{x}) = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, so

$$\text{ref}_L(\vec{x}) = 2\,\text{proj}_L(\vec{x}) - \vec{x} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}.$$

Let $a = 2u_1^2 - 1$ and $b = 2u_1u_2$. Note the sum of the diagonal entries

$$(2u_1^2 - 1) + (2u_2^2 - 1) = 2(u_1^2 + u_2^2) - 2 = 0$$

since $\vec{u}$ is a unit vector. Thus, $2u_2^2 - 1 = -a$, so the matrix has the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ as required.

Finally, $a^2 + b^2 = (2u_1^2 - 1)^2 + (2u_1u_2)^2 = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2u_2^2$. Rewrite the final u_2^2 as $(1 - u_1^2)$ and this simplifies to $a^2 + b^2 = 1$. (Alternatively, note that $4u_1^4 - 4u_1^2 + 1 + 4u_1^2u_2^2 = 4u_1^2(u_1^2 - 1 + u_2^2) = 0$.)

17. First, solve

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

That is, we want to row-reduce the augmented matrix $\left[\begin{array}{cc|c} (a-1) & b & 0 \\ b & -(a+1) & 0 \end{array}\right]$.

Dividing the first row by $(a-1)$ gives $\left[\begin{array}{cc|c} 1 & \frac{b}{a-1} & 0 \\ b & -(a+1) & 0 \end{array}\right]$.

Subtracting b times the first row from the second and getting a common denominator in the second row,

$\left[\begin{array}{cc|c} 1 & \frac{b}{a-1} & 0 \\ 0 & \frac{1-(a^2+b^2)}{a-1} & 0 \end{array}\right]$.

Since $a^2 + b^2 = 1$, the second row is all zeroes, which gives us infinitely many solutions of the form $\begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$.

We only need one solution, so let $t = 1$, for example. Then $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$. (Note this is nonzero unless $a = 1$ and $b = 0$. In this special case, let $\vec{v} = \vec{e_1}$ and $\vec{w} = \vec{e_2}$ instead.)

We could compute $\vec{w}$ by the same technique, or we could note that $\vec{v}$ and $\vec{w}$ are going to end up being perpendicular, and use the equation $\vec{v} \cdot \vec{w} = 0$ to solve for

$\vec{w}$. (If you use the latter technique, be sure to verify that $A\vec{w} = -\vec{w}$. If you use the former technique, be sure to verify that $\vec{v} \cdot \vec{w} = 0$.) This gives $\vec{w} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$.

Define the line L to be the span of the vector $\vec{v}$. Then we may write any $\vec{x}$ in $\mathbb{R}^2$ as

$$\vec{x} = \vec{x^{\parallel}} + \vec{x^{\perp}} = c\vec{v} + d\vec{w}$$

for some scalars c and d.

Then

$$T(\vec{x}) = A(c\vec{v} + d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x^{\parallel}} - \vec{x^{\perp}} = \text{ref}_L(\vec{x}).$$