This collection of problems is intended as a challenge, as a way of examining some of the more interesting uses of what we've seen, and as an introduction to some generalizations of some of these ideas that you might encounter in later courses. Some are straight-forward, while some are broad and only amorphously defined. Some are easy, while some are quite difficult and some may be impossible (but I don't think so). Don't worry if you can't do all of them: that was my intention. You do not need to attempt these problems if you don't want to. None of these will be collected. These haven't been thoroughly proofread, but I'm pretty sure that all of the statements in the problems are true. If not, correct them before proving them.

1 Chapter 6 Challenge Problems

Book

Section 6.1: 58 (partial answer: (a) can't be done, (b) can be done), 62-66 (answer to #66b: basis is $\{\det(A)\}$, so the dimension is 1)
Section 6.2: 44, 45 (use the Laplace expansion defined on p. 270 on the first column), 48, 49 (start by doing this with an upper-triangular matrix, then modify it to make the entries below the diagonal nonzero too), 50 (answer: 1), 55 (this is used to define the determinant at the graduate level)

More problems

1. Use QR factorization and our theorems about determinants to show that $|\det(A)|$ is the volume of the parallelepiped formed by the columns of A.

2. Recall the Vandermonde determinant from #31 in Section 6.2.

   Use Vandermonde determinants to show that the linear transformation

$$T(f) = \begin{bmatrix} f(a_0) \\ \vdots \\ f(a_n) \end{bmatrix}$$

   from $\mathcal{P}_n$ to $\mathbb{R}^{n+1}$ is an isomorphism for any distinct scalars $a_0, \dots, a_n$.

3. Use Vandermonde determinants to show that there is a unique polynomial of degree $\leq n$ that passes through the data points $\{(x_1, y_1), \dots, (x_n, y_n), (x_{n+1}, y_{n+1})\}$ if and only if no two points have the same x-coordinate.

4. Use Vandermonde determinants to show that given any $n+1$ distinct real numbers $t_0, \dots, t_n$ and any $n+1$ real numbers $y_0, \dots, y_n$, there is a unique function of the form $f(x) = a_0 e^{t_0 x} + \cdots + a_n e^{t_n x}$ that satisfies the constraints $f(0) = y_0, f'(0) = y_1, \dots, f^{(n)}(0) = y_n$.

5. Given n vectors $\vec{v_1}, \ldots, \vec{v_n}$ in $\mathbb{R}^N$ (for some fixed N), we define the *Gram matrix* $\mathrm{Gram}(\vec{v_1}, \ldots, \vec{v_n})$ to be the $n \times n$ matrix whose ijth entry is $\vec{v_i} \cdot \vec{v_j}$. Then, we define *Gram determinant* or *Gramian* $G(\vec{v_1}, \ldots, \vec{v_n})$ to be the determinant of $\mathrm{Gram}(\vec{v_1}, \ldots, \vec{v_n})$. Show that:
   (a) For any vectors $\vec{v_1}, \ldots, \vec{v_n}$ in $\mathbb{R}^N$, $G(\vec{v_1}, \ldots, \vec{v_n}) \geq 0$.
   (b) More generally, $\mathrm{Gram}(\vec{v_1}, \ldots, \vec{v_n})$ is *positive semi-definite*, which means that
   $$\vec{v}^{\mathrm{T}}\mathrm{Gram}(\vec{v_1}, \ldots, \vec{v_n})\vec{v} \geq 0$$
   for all vectors $\vec{v}$ in $\mathbb{R}^n$. (Compare this with the definition of positive definite in #6 of the Chapter 5 Challenge Problems.)
   (c) $\mathrm{Gram}(\vec{v_1}, \ldots, \vec{v_n})$ is positive definite if and only if $G(\vec{v_1}, \ldots, \vec{v_n}) > 0$ if and only if $\vec{v_1}, \ldots, \vec{v_n}$ are linearly independent.
   (d) $G(\vec{v_1}, \ldots, \vec{v_n})$ is the square of the volume of the parallelepiped formed by $\vec{v_1}, \ldots, \vec{v_n}$.
   (e) The $n = 2$ case of parts (a) and (c) above is (after rearrangement) something we saw in Chapter 5. What?

   (To this end, it's useful to first prove the following:

   Let A be a (real) symmetric $n \times n$ matrix which is positive definite and let $\vec{v}$ in $\mathbb{R}^n$ be any vector. Then $(A\vec{v}) \cdot \vec{v} = 0$ if and only if $A\vec{v} = \vec{0}$.)

6. Let V be an inner product space. Generalize the above problem to V, using the inner product on V in place of the dot product.

2 Chapter 7 Challenge Problems

Book

Section 7.1: 44, 45, 47, 48
Section 7.2: 14, 29, 30, 31, 33, 36
Section 7.3: 38
Section 7.4: 56, 57, 69
Section 7.5: 30-32

More problems

1. We've seen that finding the eigenvalues requires us to factor the characteristic polynomial. However, in practice, this polynomial may be hard to factor and we may need to approximate the roots numerically (e.g., $\lambda \approx 1.2837193$). If

we only have an approximation to the eigenvalue, our technique for finding the corresponding eigenvectors needs to be changed too. If we have an approximation to the eigenvalue, how can we find an approximation to the eigenvector? How good do these approximations need to be in order to be find approximate solutions for the kinds of problems we're interested in (e.g, dynamical systems, calculating large powers of matrices)?

2. What is the relation between the eigenvalues of A and the eigenvalues of $\mathrm{rref}(A)$? Are they always the same? Are certain particular eigenvalues always the same? How do the algebraic and geometric multiplicities change?

3. Use the previous problem to give a direct proof of: "0 fails to be an eigenvalue of A if and only if $\mathrm{rref}(A) = I_n$." (We saw this in class as part of Summary 7.1.5, but used kernels in our proof there. Do this without referencing kernels.)

4. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the $n \times n$ matrix A, repeated according to algebraic multiplicity and listed in terms of decreasing absolute value. Describe the long term behavior of the dynamical system $\vec{x}(t) = A^t \vec{x_0}$ in terms of the eigenvalues of A. (*Hint:* Usually, it's enough to look at only λ_1, but you'll need to do a bit more if $-\lambda_1$ is also an eigenvalue of A.)

5. Let V be a finite dimensional vector space and T a linear transformation from V to V. State and prove an analogue of Summary 7.1.5 (p. 305 in the book).

6. Given a polynomial
$$f(x) = a_m x^m + \cdots + a_1 x + a_0$$
from $\mathbb{R}$ to $\mathbb{R}$, consider the related "matrix polynomial" (which we'll call by the same name as f)
$$f(M) = a_m M^m + \cdots + a_1 M + a_0 I_n$$
from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$. Let A be an $n \times n$ matrix. Let $\vec{v}$ be an eigenvector of A with associated eigenvalue λ. Let $g(x)$ be a polynomial and $g(M)$ be the associated matrix polynomial. Show that $\vec{v}$ is an eigenvector of $g(A)$ with eigenvalue $g(\lambda)$. That is, that $g(A)\vec{v} = g(\lambda)\vec{v}$. Compare with #1-4 in Section 7.1.

7. Let $f_A(\lambda)$ be the characteristic polynomial of a matrix A and $f_A(M)$ be the associated matrix polynomial of f_A (as in problem #6 above). Use #6 to show that $f_A(A)$ is not an invertible matrix. (*Hint*: Show that 0 is an eigenvalue of $f_A(A)$.)

8. We can say something even better in the above problem, by proving the Cayley-Hamilton Theorem: Let $f_A(\lambda)$ be the characteristic polynomial of a matrix A and $f_A(M)$ be the associated matrix polynomial of f_A (as in problem #6 above).

Then $f_A(A) = 0$ (where 0 denotes the zero matrix). (*Hint:* One way of doing this is presented in problem #54 of Section 7.3. Try it on your own first, since there are other ways of approaching the problem. If you have trouble doing it in general, try proving it for the special case where A is diagonalizable, using #70 of Section 7.4.)

9. We say that an $n \times n$ matrix is a *transition matrix* if all of its entries are nonnegative and if the sum of the entries in each column is 1. (Equivalently, if for each column $\vec{v}$, we have $\vec{v} \cdot (\vec{e_1} + \cdots + \vec{e_n}) = 1$.) (See #25, 30, and 31 in Section 7.2 and #30 of Section 7.5 for a closely related concept, the *regular transition matrix*, in which we add the extra restriction that no entry is 0.) Give a real-world interpretation of transition matrices and use this interpretation to explain the results in Section 7.2, #30 and 31 and Section 7.5, #30 in real-world terms. (*Hint:* #4 above will help.)

10. As we defined them, discrete linear dynamical systems seem to be unable to model phenomena that involve more than one recursive term, or which involve adding a constant. For example, we'd have trouble modeling the system given by

$$\begin{aligned} a_{n+2} &= 2a_{n+1} + 3a_n + 4b_{n+1} + 5b_n + 6 \\ b_{n+2} &= 7a_{n+1} + 8a_n + 9b_{n+1} + 10b_n + 11 \end{aligned}$$

with initial conditions $a_0 = 12, a_1 = 13, b_0 = 14, b_1 = 15$. (The specific numbers aren't important Replace them with arbitrary constants if you'd like.)

However, despite this impression, we *can* model this as a dynamical system. Do #42, 45, and 48 in Section 7.3 and use the same ideas to set up the above as a dynamical system.

Hint: You'll probably want to define $\vec{x}(t) = \begin{bmatrix} a_{t+2} \\ a_{t+1} \\ a_t \\ b_{t+2} \\ b_{t+1} \\ b_t \\ 1 \end{bmatrix}$.

(Or, since we're most interested in a_{t+2} and b_{t+2}, you can rearrange the components so that these two come first, if you wish.) Note that as in our examples in class, the switch from n to t is purely notational.

11. (the pièce de résistance) We've seen that we can also write sequences as functions, so we could write $a(t)$ instead of a_t. We'll do that in this problem, as we're going

to use subscripts for something else. Thus, the system in the above problem can be rewritten as

$$\begin{aligned} a(n+2) &= 2a(n+1)+3a(n)+4b(n+1)+5b(n)+6 \\ b(n+2) &= 7a(n+1)+8a(n)+9b(n+1)+10b(n)+11 \end{aligned}$$

and if also rename our sequences a_1 and a_2 (so, $a_1(n) = a(n)$ and $a_2(n) = b(n)$), this becomes

$$\begin{aligned} a_1(n+2) &= 2a_1(n+1)+3a_1(n)+4a_2(n+1)+5a_2(n)+6 \\ a_2(n+2) &= 7a_1(n+1)+8a_1(n)+9a_2(n+1)+10a_2(n)+11. \end{aligned}$$

Suppose we have sequences $a_1(t), \ldots, a_m(t)$ for some positive integer m and scalars $p_1, \ldots, p_m$, $d_1 \ldots, d_m$, and c_{hij} for $1 \leq h \leq m$, $1 \leq i \leq m$ and $0 \leq j < p_i$ and consider the system

$$\begin{aligned} a_1(t+p_1) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{1ij}a_i(t+j)\right) + d_1 \\ a_2(t+p_2) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{2ij}a_i(t+j)\right) + d_2 \\ &\vdots \\ a_k(t+p_k) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{kij}a_i(t+j)\right) + d_k \\ &\vdots \\ a_m(t+p_m) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{mij}a_i(t+j)\right) + d_m \end{aligned}$$

(with some appropriate initial conditions.)

It's more convenient to have all of the entries on the left line up, so we'll rewrite this as

$$\begin{aligned}
a_1(t) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{1ij}a_i(t-p_1+j)\right)+d_1 \\
a_2(t) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{2ij}a_i(t-p_2+j)\right)+d_2 \\
&\vdots \\
a_k(t) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{kij}a_i(t-p_k+j)\right)+d_k \\
&\vdots \\
a_m(t) &= \left(\sum_{i=1}^{m}\sum_{j=0}^{p_i-1} c_{mij}a_i(t-p_m+j)\right)+d_m.
\end{aligned}$$

Define $\vec{x}(t)$ and a matrix A so that the above is given by the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$. What are the dimensions of $\vec{x}(t)$ and of A? Set up the initial conditions above explicitly, and use them to find $\vec{x}(0)$. Can you say anything in general about the form of A? (*Hint*: think block matrices) Can you say anything in general about the eigenvalues of A? (*Hint*: start with the special case in which A is *block upper-triangular* (meaning that A can be written as a block matrix in which the all blocks below the "block diagonal" are the zero matrix, so for example $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ is block upper-triangular with 2×2 blocks) by generalizing Section 7.2, #14. What does the condition that A is block upper-triangular tell you about the coefficients c_{hij}? *Note*: The block upper-triangular case is hard but doable; I'm not sure what the answer is in the general case.)

(To help you get started, I'll give you $\vec{x}(t)$. If you want to find it on your own, stop reading now.)

$$\vec{x}(t) = \begin{bmatrix} a_1(t) \\ a_1(t-1) \\ \vdots \\ a_1(t-p_1) \\ a_2(t) \\ \vdots \\ a_2(t-p_2) \\ \vdots \\ a_m(t) \\ \vdots \\ a_m(t-p_m) \\ 1 \end{bmatrix},$$

so that $\vec{x}(t)$ is in $\mathbb{R}^s$ where

$$s = \left(\sum_{i=1}^{m} (p_i + 1) \right) + 1 = \left(\sum_{i=1}^{m} p_i \right) + m + 1.$$

Or, you can rearrange the order of these components if you wish (especially if doing so gives your matrix a nicer form!).